The Arithmetic Teacher

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Fundamental Issues in the Teaching of Elementary School Mathematics
MARSHALL H. STONE

What Numbers Mean to Young Children
AGNES AND ETHEL GUNDERSON

Pre-school Children's Familiarity with Measurement
O. L. DAVIS, BARBARA CARPER, AND CAROLYN CRIGLER

Observations on the Contemporary Scene
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A Journal of
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE ARITHMETIC TEACHER

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The National Council of Teachers of Mathematics

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THE ARITHMETIC TEACHER

Volume VI Number 4

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Fundamental Issues in the Teaching of Elementary School Mathematics*

MARSHALL H. STONE
The University of Chicago

THE COMMITMENT OF OUR NATION to an intensive scientific and technological development puts heavy pressure on our schools and colleges to produce an increased number of graduates well trained to pursue trades, professions, and scientific or academic careers dependent upon mathematical skills and abilities. While this pressure is felt especially in high schools, colleges and universities, its effects are reaching the grade-schools as well. From the stream of students flowing through our educational system we must obtain a greater proportion willing to pursue mathematical studies to more or less advanced levels, and we must in particular guarantee that among the students leaving the grade-schools an increased number will have a taste and inclination for mathematics sufficiently strong for them to study further mathematics in high school. This necessitates an improvement and enrichment of the mathematical elements in grade-school education.

These problems of education are not an exclusively American phenomenon. It is interesting to note that the Organization for European Economic Cooperation, because of its aim to increase the productivity and well-being of Europe's economy, has decided that it must now explore means for improving mathematical and scientific training in European schools.

Specific Problems Faced

If we follow the flow of students through our own schools we note that we have a number of specific problems—to provide for greater proportions of the flow to continue in contact with mathematical studies after each passage of a traditional check-point (graduation from grade school, from high school, from college and so on), to ensure a greater feedback of mathematics and science teachers into our universities, colleges, high schools, and grade schools, and to improve and accelerate the mathematical training offered at all levels. As the mathematical requirements of science and technology become more varied and complex, we must avoid prolonging for our students their period of apprenticeship, finding ways to streamline and accelerate wherever possible their mathematical preparation. From these considerations there emerge five fundamental issues in the teaching of elementary mathematics: how to change the content of grade-school mathematics so that it is richer, better adapted to current needs, and better suited to prepare grade-school pupils for their entry into high school mathematics

* This is a summary of Professor Stone's presentation to a group considering elementary school Mathematics under the sponsorship of the School Mathematics Study Group of Yale University and held at Chicago on February 12 and 13, 1959.

courses in algebra and geometry; how to take advantage of special methods and materials which will enhance the effectiveness and appeal of our class-room teaching; how to define in an adequate statement the full objectives of elementary training in mathematics; how to apply the psychologists' discoveries in the fields of child development and concept-formation to the elaboration of an elementary curriculum more closely adjusted to the unfolding capabilities of our grade-school children; and, finally, how to recruit and train better elementary school teachers of mathematics.

Content and Method

The content of our courses in arithmetic may not require much change, but there is good ground for thinking that what we teach should be begun earlier and not drawn out so long. On the other hand there is much intuitive geometry which can appropriately be introduced into grade school courses. If this were done, the mathematical training of grade-school pupils would be notably enriched and would be rendered very much more effective as preparation for high school mathematics. In fact, the teaching of high school geometry could be very much improved and strengthened if students were given a thorough preliminary experience in intuitive geometry, going a good way beyond the fragmentary treatment of mensuration usually coupled with arithmetic as a convenient source of clothed arithmetic problems.

We know that a number of very special and effective methods have been developed for teaching elementary arithmetic. Several of these require the use of special materials or apparatus, sometimes of a rather elaborate and expensive kind. It is important that we take all the advantage of these teaching devices that we possibly can. Having in mind their use on a mass-scale, we need to test, criticise, simplify, and adapt them to a point where it is practical to introduce them generally in our grade-school classrooms. A number of these methods and devices will be discussed in this conference, so that members can form their own judgments as to the relevance and value of these remarks.

Content and methods both depend in great measure upon the objectives sought in teaching elementary mathematics. For practical and cultural reasons, we have no choice but to teach a good deal of arithmetic and intuitive geometry to our boys and girls. However, it is not so generally accepted that our objective must be the double one of developing purely technical skills (e.g. in the art of computation) and of preparing the way towards mathematical insight into the relations which give arithmetic and elementary geometry their characteristic structures. These contrasting aims are often distinguished by the adjectives "concrete" and "abstract." What we really mean by abstraction is the recognition of *patterns*, as in a geometric configuration, which have a special significance for the mathematical structure about which something is being learned. It is essential to recognize that technical skill and insight go hand in hand in mathematics—each strengthens the other, and the mathematician who lacks either one cannot expect to attain his maximum development as a mathematician. This is why it is so very important to have both aims in mind, even at the grade school level.

The organization of the topics which we wish to teach in grade-school mathematics into a program designed to give a steadily increasing mastery of useful techniques and a steadily expanding pattern of essential insights cannot be undertaken without precise and firmly established knowledge of the psychology of the growing child. While psychologists can provide much more information about the unfolding of the child's mind than they could twenty-five or fifty years ago, it seems clear that they are only on the threshold of an understanding of the problems of mental development and concept formation. Here then is an area where whatever we do now must be frankly tentative, where we must expect to do a good deal of fundamental research before we can achieve the results we should aim at on the

pedagogical side. There is no doubt that true progress in this fascinating field can be made only through the cooperation of all those concerned—of mathematicians, psychologists, and teachers.

Training of Teachers

If we wish to improve the teaching of grade-school mathematics in these rather sophisticated and fundamental ways, it will be necessary for us to give much better training to the future teachers of elementary mathematics. Because the turnover of teachers in elementary schools is very high, improved training can quickly have a large-scale effect on what is accomplished there. Even without any attempt to modify the curriculum at all, there is good reason to give our elementary school teachers better training in mathematics. It is only too certain that today's mathematically ill-prepared teachers, many of whom are ill-disposed toward the subject of mathematics, are infecting too large a number of our boys and girls with an enduring fear and hatred of mathematics, which can rarely be overcome later on in high school. The new mathematics training which may be developed for future grade-school teachers should give them an appreciation and understanding of the subjects they may be expected to teach and as a result should inspire them with a certain degree of respect and admiration for mathematics, if not a real liking of it.

Summary

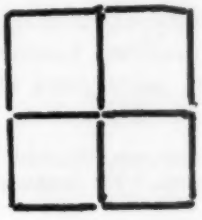
It seems to me that something can be done without delay or postponement in regard to each of the five issues which I have singled out for consideration. We can get together groups to discuss objectives and the desiderata in teacher training, other groups to explore the possibilities for fruitful investigation of the psychological factors which underlie the teaching of elementary mathematics, and still other groups to prepare and test textbooks, workbooks, teacher manuals, special methods, and special apparatus. Whatever we may do at the beginning can only be the starting point for a long-range program. The need for such a long-range program dealing with elementary mathematical instruction is going to become more compelling and more evident the more we struggle with the problems at the high-school and college levels. It is of the greatest importance that we make a start now!

Editor's Note. It is most heartening to note the concern of the mathematicians as represented by Professor Stone in the problems of mathematics instruction in the elementary schools. This interest stems from a genuine concern with the nature of the mathematics to be taught, how it is to be learned, and by whom it is taught. It is also interesting to note that many "educators" are beginning to be concerned with the future of this country and its dependence upon a technological development in which mathematics provides many important links and steps. While this group may not understand the situation fully, they sense something about it and that is a good first step toward a better education of the boys and girls in our schools. As Professor Stone says, we need the cooperation of the mathematicians, the psychologists, and the people concerned with educating our youth. "It is of the greatest importance that we make a start now!" for the improvement of the mathematics program from the kindergarten through the graduate school.

Stick Puzzle

If you have 12 sticks of equal length arranged as in the diagram below, show how you can make the following rearrangements.

1. Remove 4 sticks and leave 2 squares.
2. Remove 4 sticks and leave 1 square.
3. Change 3 sticks and have 3 squares.
4. Remove 2 sticks and leave 2 squares.



What Numbers Mean to Young Children

Number Concepts Compared With a 1940 Study

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TWENTY YEARS AGO a study[1] was made at the University of Wyoming of number ideas or concepts held by a group of 17 pupils at the University Elementary School at the completion of the first half of the second grade. This inventory or study showed that seven-year-olds have many contacts with the use and application of number in their daily work and play. It indicated that children learn many number concepts and associate various meanings and understandings with numbers 1–12.

The thought occurs: If a similar study were made now, 20 years later, how would the findings compare with those of the earlier study? This led to making such a study with a group of 26 seven-year-olds in Grade Two in the Willets Road School, East Williston, L. I., New York, at the beginning of the second semester of the school year 1958–59.

The same technique was used in this study as in the earlier one. The teacher asked the pupils to tell all they knew about the numbers, 1–12. Questions as: What does *five* mean? What is *five?* What does *five* make you think of? How can you show *five?* Do you know anything else about *five?* All work was oral, the teacher jotting down whatever responses the children made—most of these were in statement form.

Arithmetic work previous to the inventory had been with concrete and semi-concrete materials, using counters to show groupings and to solve problems. Learning and using measures: feet, yards, pints, quarts, dozen, hours, days, weeks, months, and years. Making a calendar for each month. Solving problems in addition and subtraction with sums and minuends to 10. Counting and writing numbers to 100, Roman numbers to XII. A chart with numbers to 100 used for meaning and sequence and referred to as our number system. Adding numbers to 10, as 10 and 6, 10 and 3, etc., Workbook for Grade Two. The number facts taught as related numbers

3	4	7	7
+4	+3	−3	−4
7	7	4	3

Fraction concepts of halves, fourths, sixths, thirds, and eighths. This content or subject matter is similar to that noted in the 1940 (Wyoming) study with this exception: in the earlier study pupils had had no work with fractions or fraction concepts. In the recent study the teacher had devoted one week, in early February, to the teaching of fraction concepts. A description of this work[2] is reported elsewhere. This work with fraction concepts preceded the taking of the inventory.

TABLE I

Number Concepts

ONE

*One (word)
*1 (figure)
1 is an odd number
*1 is the first number in the number system
*1 object (book, box, pencil, etc.)

[1] Agnes G. Gunderson, "Number Concepts Held by Seven-Year-Olds," *The Mathematics Teacher*, 33: 18–24, January 1940.

[2] Agnes G. Gunderson and Ethel Gunderson, "Fraction Concepts Held by Young Children," THE ARITHMETIC TEACHER 4: 168–173, October, 1957.

1 is my lucky number
We have one universe
We have one beautiful outdoor environment
We have lived on only one planet
We have one moon going around our planet
You have one life; you should take care of it
Do one thing at the time
You are only allowed to have one wife at the time
There is one Dr. Doolittle
Pray to only one God
*One-year-old baby
*One month until my birthday
One more year and we will be in Third Grade
*One more week we are moving to New York City
*Jack's baby brother is one month
*It is one minute after 2 o'clock now
*One o'clock in the morning I sleep
George Washington was the first president
*January 1 is the first day of the year
*Last year I was in First Grade
Here is one dollar

$0+1=1$ $2-1=1$

TWO

*Two is the word for the number 2
2 is an even number
*2 (figure)
*II (Roman number)
*2 objects
*2 arms, 2 legs, 2 feet, etc.
Two is a homonym
Channel 2 is a good channel
*Our easel has two sides on which we can paint
We have two poles on the globe—South Pole and North Pole
Two knobs on most doors
You have two parents unless there is a divorce
*We have two days in our week-end
*My baby sister Amy is two years old
*We are in Grade Two
Moses split the Red Sea in two parts
*I fell down twice on the ice
*It is 2 minutes after 9 o'clock A.M.
*Monday is the second day of the week
November is second to the last month in the year
B is the second letter in the alphabet
2 tens make 20
*2 halves make a whole
*2 nickels make a dime
2 pints make a quart
*2 counters measure 2 inches
200 pennies make 2 dollars
2 eighths make a fourth

$1+1=2$ $5-3=2$ $4-2=2$ $100-98=2$

THREE

*Three (word)
*3 (figure)
*III (Roman number)
3 is an odd number
*3 objects
*My family has three children
*We had a three-layer cake
*RUN has 3 letters
*Three Blind Mice
*I liked the story of The Three Bears
*At 3 o'clock we are home from school
X is the third letter from the last in the alphabet
C is the third letter in the alphabet
*Tuesday is the third day of the week
The middle finger is the third in line
March is the third month of the year
*3 feet make a yard
*3 counters measure 3 inches
3 sixths make a half
3 thirds of a pie make a whole pie
3 parts would be 1 third each
*Three 3's make 9

$8-5=3$ $5-2=3$ $9-6=3$
$6-3=3$ $1+2=3$ $2+1=3$

FOUR

*Four (word)
*4 (figure)
*IV (Roman number)
*4 objects
*Four is a homonym as four, for,
*The number before 5 is 4
4 is an even number
*4 is the fourth number in our number system
On Saturday Channel 4 has basketball
4 is my lucky number
In the alphabet D is the fourth letter
A lucky clover has 4 leaves
*A square has 4 sides
*The word *cent* has 4 letters
*My dog has 4 legs
I have 4 dollars in my bank
A human being has 2 legs and 2 arms and that makes 4
When we buy butter there are 4 sticks in the pound
There are 4 second grades in our school
Without your thumb you have 4 fingers on a hand
*My sister is 4 years old
In 4 years it will be 1963
Every four years we elect a president
*Four o'clock is Junior Town on TV
My pencil is 4 inches
*It is 4 minutes to 9 now
The fourth finger is the one you wear a ring on
On July 4th this year, 1959, the new flag with 49 stars goes up
*Wednesday is the fourth day of the week
April is the fourth month of the year
*Two pairs of shoes are 4 shoes
*In fractions you need 4 fourths to make a whole
4 eighths makes a half

$7-3=4$ $10-6=4$ $14-10=4$ $1+1+1+1=4$
$2+2=4$ $3+1=4$ $5-1=4$

FIVE

*Five (word)
*5 (figure)
5 is an odd number
*5 objects

*5 is the fifth number in the number system
*We count by 5's
5 is really 5 ones
*We usually have 5 schooldays a week
*5 pennies make a nickel which is 5 cents
We traveled in New England; we traveled in 5 states
*There are 5 fingers on one hand
If you cut a pie in 5 pieces that would be fifths
*At 5 o'clock I watch Abbott and Costello
*When you are 5 you are a pretty big boy
My hand is 5 inches long
Five years is kindergarten age
Five more minutes it will be 10 o'clock
Tom Thumb is a little above 5 inches
Michael is the fifth child from the lectern
*My birthday is January 5th
E is the fifth letter in the alphabet
*May is the fifth month in the year
If you are 15 and you take away 10 years you'd be 5 years old
*3 boys and 2 girls make 5 children
*1 girl and 2 boys and parents make 5 in a family

$4+1=5$ $7-2=5$ $2¢+3¢=5¢$ $9-4=5$

SIX

*Six (word)
*6 is a Hindu-Arabic number
*VI is a Roman number for 6
*6 objects
6 is an even number
*5 comes before 6
*7 comes after 6
WONDER has 6 letters
Channel 6 doesn't show a picture on our TV
Some insects have 6 legs
*After you are 6 your are a big boy
*Ellen is 6 years old
6 minutes before 12 noon
*3 feet is a yard so 6 feet are 2 yards
The average man is about 6 feet tall
*When you are 6 you are in first grade
*Our family eats dinner at 6
6 inches is half a foot
*F is the sixth letter of the alphabet
*Sixth Grade is the last grade in this school
Three pairs of mittens makes 6
Two 3's make 6

*3 dozen and 3 dozen is 6 dozen
Is you cut a pie in sixths you have this

1 sixth of a pie is enough for me

$9-3=6$ $10-4=6$ $8-2=6$
$7-1=6$ $12-6=6$ $3+3=6$
$2+2+2=6$ $4+2=6$ $2+4=6$
$1+5=6$ $5+1=6$

SEVEN

*Seven (word)
*7 (figure)
*7 objects
7 is an odd number
*6 comes before 7
7 ones make 7
On the back of Mickey Mantle's shirt is number 7
Good shows are on Channel 7
We cross the Seven Seas
Seven Voyages of Sindbad
*MICHAEL has 7 letters
*I am 7 years old
*Number 7 is the seventh number in the number system
*A week has 7 days
*Seven more years I will be as old as my brother
*We usually take 7 quarts of milk a day
*Seven minutes until 11 o'clock
From kindergarten it takes 7 years to get out of this school
700 pennies is 7 dollars
Some people have a shoe size 7
*July is the seventh month of the year
*Saturday is the seventh day of the week
1 nickel and 2 cents make 7 cents

$6+1=7$ $5+2=7$ $4+3=7$ $3+4=7$
$3+2+2=7$ $8-1=7$ $14-7=7$ $9-2=7$
$10-3=7$ $11-4=7$

EIGHT

*Eight (word)
*8 (figure)
*8 is the eighth number in our number system
*8 objects
An octopus has 8 legs
Charlotte (spider in book CHARLOTTE'S WEB) has 8 legs
There are 8 planets besides the earth
We can't get a picture on Channel 8
*I am already 8 years old
When a Brownie is 8 years old she is happy
You have to be 8 to be a Cub Scout
*8 o'clock is bedtime
It takes a jet about 8 hours to go to France from here (Long Island)
If you went to the store and you wanted a half pound of something you could say, "Give me 8 ounces"
Each of my counters measures 1 inch so it takes 8 counters to make 8 inches
*The 8th of March is on a Sunday
Two pies cut in fourths make 8 pieces

If you cut a pie in eighths like this you have 8 pieces

$5+3=8$ $4+4=8$ $16-8=8$ $2\times4=8$ $7+1=8$

NINE

*Nine (word)
*9 is Hindu-Arabic
*IX is Roman number
This is the year 1959
*9 objects
*Counting by 3's
Counting 1, 3, 5, 7, 9
*9 is the ninth number in the number system
*8 comes before 9
9 boys on a baseball team
9 planets in the solar system
*My friend is 9 years old
9 feet is long
*Some people go to bed at 9 o'clock
9 inches
September, the ninth month
I is the ninth letter of the alphabet

3+3+3=9 6+3=9 7+2=9 2+7=9
4+5=9 5+4=9 10−1=9 18−9=9
11−2=9 100−91=9

TEN

*Ten (word)
*10 (figure)
*In Roman numbers 10 is X
*10 is the tenth number in the number system
10 is an even number
*10 comes after 9
Our whole number system is based on tens
*Counting by 2's to 10
*10 objects
*There are 10 fingers on two hands
There may be 10 planets but we are sure of 9
*I bought a microscope for 10 dollars
Once my father got something on Channel 10
RECREATION has 10 letters in it
*We usually have recreation at 10 o'clock
There are 10 months in the school year
When you are 10 years old you are in 5th grade unless you skipped
This book measures 10 inches
The Long Island Railroad is usually 10 minutes late, my father said
*10 cents is a dime
When it is 10 degrees it is real cold
Sometimes in some parts of the world like Alaska it is 10 degrees below zero
It takes the bus about 10 minutes to come from my home to school
10 acres is a lot of land
Some trees are 10 feet tall
*May 10th is Shaarie's birthday
I weighed 10 ounces and 6 pounds when I was born
*2 nickels makes 10 cents
If you had 2 pies and you cut one in fourths and the other in sixths you would have 10 pieces of pie

5+5=10 6+4=10 20−10=10

ELEVEN

*Eleven (word)
*11 (figure)
*XI is 11 in Roman numbers
Eleven rhymes with seven
*11 objects
11 is made up of 1 ten and 1 one
11 is an odd number
11 is my lucky number
*11 is 1 less than 1 dozen
Channel 11 has sports
Mickey Mantle hit 11 home runs
*My address is 11 Field Lane
11 inches is almost a foot
*At 11 o'clock A.M. we usually read
My brother gets out of kindergarten at 11 o'clock
Shock Theater is on at 11 o'clock
My baby brother is 11 months old
11 feet is the height of the class room
11 hours is a long time, 11 minutes is not so long
*My brother is 11
*November is the eleventh month
*The eleventh grader is a junior in high school

5+6=11 2+8+1=11 8+3=11 3+8=11
6+5=11 7+4=11 4+7=11

TWELVE

*Twelve (word)
*12 is Hindu-Arabic
*Roman number 12 is XII
*12 is the 12th number in the number system
*12 objects
"Time of Wonder," the name of our mural is 12 letters
*A dozen things is 12
Most kids are in junior high school when they are 12
*12 years old
When a girl is 12 she usually gets a boy friend
*12 o'clock is our lunch time
*There are 12 hours from night until today
*Our poetry folder is 12 inches long
Lincoln's birthday comes the 12th day of February
L is the 12th letter of the alphabet
*12 is 10 and 2
I can buy 2 five-cent candy bars and 2 pieces of gum for 12 cents
One dime and 2 cents make 12 cents
I can buy 4 snack milks for 12 cents
I can buy 12 bubble gums for 12 cents
We have 6 windows in this room. Each window has 2 curtains. That makes 12
*Three 4's are 12
*Four 3's are 12
*6 and 6 are 12

---

* Similar comments were listed in the 1940 (Wyoming) study.

TABLE II

Fraction Concepts

ONE HALF

You could say the earth is divided in half by the equator; there is the north half and the south half
*Our library table is in two parts, each part is a half
Split a day in half and you have 12 hours
*From me down to Paul, the room is cut in half
*You have 10 counters; you divide them in half, you then have 5 counters
I can eat a half pound of meat
*Sometimes there is a half moon in the night sky
A half is to split something in two equal parts
*Two halves make a whole
Two half hours make a whole hour
*I can eat a half pie
2 fourths make a half
4 eighths make a half
3 sixths are a half pie
You can cut anything in half
If you cut a fourth in half you have 2 eighths
*A half quart of milk
Two sides to the world; one half is light, one half is dark
*A half dollar is 50 cents
A nickel is half of a dime
You can split a half dollar in half and you have 2 quarters
In some foreign money you have a half penny
By Wednesday noon half the week is gone
*We can buy a half pound of nuts
I weigh 65 and one half pounds
Monday divide in half: Mon-day
*If you have 4 pencils, half would be 2

10¢−5¢=5¢

ONE FOURTH

*If you had a pie cut in fourths you'd have 4 pieces
*A fourth is littler than a third
A fourth is bigger than a sixth
If I wanted 1 fourth of an English muffin, first I'd halve it in the center, then cut each piece in half
It takes 2 eighths to make a fourth
If you divide an hour in fourths you have 15 minutes or a fourth of an hour
*4 fourths make a whole
Three months is 1 fourth of a year
If I cut a pie in fourths and if I eat 1 piece I'd have 3 fourths of the pie left
In a fourth of an hour it will be 10 o'clock
Friends came over; we cut a pie in eighths. We ate 6 pieces; there were 2 eighths or 1 fourth of the pie left
One fourth of the moon is about like this (sketch)
I asked my sister for a fourth of the cake but she gave me 1 half of the fourth which is, as you know, an eighth

* Similar comments were listed in the 1940 (Wyoming) study.

Put 8 children in a row

XX, XX, XX, XX

Ronnie and Richard make a fourth; Amy and Debby make a fourth; Penny and Shaarie make a fourth; Ken and Gerri make a fourth
Put 4 children in a group

x
x x
x

Each child would be a fourth, Paul, Ellen, Sally, Debby
A fourth of a banana isn't very much

ONE THIRD

One third piece of pie is big
*You have 3 pieces if you cut a pie in thirds
1 third is bigger than 1 fourth
2 sixths make a third
About 1 third of the planet is land; 2 thirds is water
*3 thirds make a whole
From January until April is 1 third of a year
20 minutes is 1 third of an hour
Sometimes we see just 1 third of the moon
On the hard-top where we play, it is 1 third of an acre

Conclusions

In comparing the responses given in this study with those of the study made 20 years ago, one notes many number concepts common to both studies. Each study reveals some ideas not shown in the other but the similarities are far greater than the differences. Both studies show that seven-year-olds understand the smaller numbers 1–12 rather well. They know: the word denoting the number, the Hindu-Arabic number, the Roman number, the number system, the cardinal and ordinal meanings of numbers. They are able to read the clock and the calendar; to compare numbers, e.g. 9 is more than 8. They know some of the easy addition and subtraction facts; they recognize number used as identification, as "Channel 4" and "Fourth Street."

The pupils in the earlier study gave a larger number of addition and subtraction facts and a larger number of words denoting quantity such as: double, coupie, twins, duet, etc., but their comments did not reflect science learning nor much of social

studies, whereas in the recent study a knowledge of science is revealed in comments about jets, planets, TV, etc. An awareness of social mores is seen in these comments: "You have two parents unless there is a divorce. When a girl is 12 she usually gets a boy friend."

It is interesting to note items 7–15 in Table I. These comments show a more specific meaning attached to *one* than do the others. Do these seven-year-olds sense the "one and only one" meaning here which adults often give to word *one* in similar comments?

This recent study shows changes in the school curriculum over the 20-year period, particularly the increasing attention given to the teaching of science—planets, television, jet planes, universe, were not mentioned in the 1940 study.

The responses given by children in this study reflect approved practices in the teaching of arithmetic which courses in teacher education, methods courses, professional magazines, modern textbooks and accompanying teachers' manuals are striving to promote. The key words of modern arithmetic teaching are *meaning* and *discovery*. The following comments indicate an understanding of the meaning of various measures:

11 hours is a long time; 11 minutes is not so long,
11 feet is the height of the classroom,
10 acres is a lot of land (this concept would vary with environment; it would be true for a child living in an urban area, to a child living on a large cattle ranch, for example, 10 acres would seem a small plot).
The hard-top where we play is 1 third of an acre;
If you wanted half a pound of something, you could say, Give me 8 ounces;
3 feet is a yard so 6 feet are 2 yards;
An average man is 6 feet tall (this from an adult's viewpoint would be an erroneous statement, but from a seven-year-old the statement is acceptable because 6 feet is more nearly the average man's height than is 5 feet.

That the number system can be meaningful to young children is seen in these comments:

4 is the fourth number in our number system;
5 is really 5 ones;
2 tens make 20;
11 is made up of 1 ten and 1 one;
Our whole number system is based on tens.

In accord with modern theories of teaching arithmetic, fraction concepts are now taught a long time before children encounter the fraction symbol in their textbooks. Comparing the 1940 study (data taken in 1939) with this study (1959): the 1940 study lists very few comparisons of fractions or fraction equivalents; those are

2 halves in 1 whole,
4 fourths in 1 whole,
2 quarters is 1 half,
1 fourth is less than 1 half,
1 half of 1 half is 1 fourth,

while the recent study lists these and many more. This greater knowledge of fractions shown in the recent study is, no doubt, a result of early teaching of fraction meanings or concepts wherein children through the use of fractional parts or cut-outs discover fraction concepts by superimposing one fractional part over another to see how they compare in size. Looking over the comments about fractions made by the pupils in this inventory one realizes this is time well spent. The comments show an understanding or insight into fractions far greater than that revealed by the pupils in the earlier study.

This inventory, like the 1940 study, is not exhaustive; the concepts listed are those which the child recalls. No doubt he would recognize many more than he is able to recall.

From the responses given above we see that seven-year-olds have a wide use for number; it is not relegated to the arithmetic class only; they use it in all their daily activities. As they are given further opportunities for discovering more and more meanings or concepts about number so will they become more efficient in their quantitative thinking.

[*Editor's Note on page 190*]

The Growth of Pre-school Children's Familiarity with Measurement

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Concepts of number and measurement are essential elements of mathematical and scientific thinking. Growth in understanding of number ideas has been studied by several investigators (1; 2; 3; 4; 8; 9). A number of studies have investigated children's understandings of certain kinds of measures (1; 3; 5; 6; 7; 8; 10; 11). Development of ideas of measurement seem to progress in a manner similar to concepts of number. Some studies have dealt with an investigation of the ideas of measurement in pre-school children (1; 5; 6; 7; 8; 10; 11), and some have investigated the growth of single measurement ideas (6; 11). MacLatchy developed an interview test for the pre-school child's familiarity with measurement (7).

Employing this instrument with 73 three-to-five year olds, MacLatchy reported, in general terms, consistent growth in children's familiarity with measures and that measures of time were more meaningful than other conventional measures (8). Spayde used the MacLatchy test with kindergarten children only and, in his conclusions, he urged many meaningful experiences for children of this age as helpful in developing adequate understandings (10).

The present study was designed to investigate the growth of familarity with measurement in children of four and five years of age.

* This study was conducted while the authors were at George Peabody College. Dr. Davis is now with the Association for Supervision and Curriculum Development in Washington, Miss Carper is at Gastonia, North Carolina, and Miss Crigler is at Whittier, California.

Method

The subjects were twenty-three four-year-old pupils enrolled in the nursery school and twenty-nine five-year-old pupils enrolled in the kindergarten of a campus demonstration school. The children were from an upper middle class population. The MacLatchy Test of the Pre-school Child's Familiarity with Measurement was administered individually to each of the children during the morning hours of the school days in February, 1958. The test took about 30 to 45 minutes to administer. The two interviewers were senior and graduate student teachers of the kindergarten group. Each teacher tested half of the children in each group. A small private room in the pre-school building was used for the interviews. Materials used in the test were kept in a box by the interviewer's side. Curious examination of the contents was discouraged but not forbidden. Few examined these materials. Enrolled in a campus school, the children were familiar with tests such as this.

Interviews were conducted in a friendly, conversational manner and recording of answers was done as inconspicuously as possible. The children in both groups seemed very much at ease and usually cooperated very well. Interest of the younger children tended to wane toward the end of the test. Only one child, from the nursery school, lost interest completely and walked away from his interview after the test was underway.

Percentages of correct responses for each item were calculated for each age group for each question of the inventory. These data are presented in Tables 1–4.

Results

Table 1 presents the percentages of correct responses for each group for each of the questions about measurement of time.

More kindergarteners knew about night and afternoon than did Nursery school children. The groups did not differ materially on answers about other parts of the day. On all questions about days of the week, the five-year-olds were superior to those in the nursery school. None of the younger children gave the days of the week in order and only a few more than one-fourth of the kindergarten children gave this information correctly.

Fewer nursery school children knew the name of the current season (winter), but, when asked to name the season "when you wear your snow suit . . . " (winter), the groups did not differ markedly. Winter and summer were identified by both groups about equally well. Understanding about spring and fall favored the kindergarten children. Referring to winter, children commented about "cold" and "snow." Comments about spring included "Easter," "flowers," and "flowers bloom, leaves come out."

Most of the children knew that people look at clocks to tell time. Only one four-year-old recognized any of the specifically requested hours on the clock. Half of the kindergarteners knew 12:00, about a fourth knew 8:00 and 6:00, but only one knew 4:30.

The four-year-olds apparently had just begun to become familiar with the calendar. Many more kindergarteners than nursery school children recognized a calendar, explained its use, pointed out a day, knew the number of days in a week, knew the current year, the name of the current month, and named other months. Yet, less than 40 per cent of the older children gave correct responses to all but the first three tasks named above. Almost none of either group identified a week or month on the calendar or named the months in order. More five-year-olds than fours knew their birthday month although over half the younger children gave the correct answer.

TABLE 1

PERCENTAGES OF CORRECT RESPONSES ON QUESTIONS ABOUT MEASUREMENT OF TIME

	N.S. $N=23$	Kgn. $N=29$
TIME		
Parts of the Day		
Morning	55.5	75.9
Night	56.5	93.1
Afternoon	21.7	58.6
Noon	0.0	6.9
Days of the Week		
Sunday	65.2	93.1
Today	8.7	55.2
Non-school days	4.3	82.8
All the days	4.3	81.2
in order	0.0	27.6
THE CALENDAR		
What is it?	47.8	72.4
What does it tell?	30.4	82.8
Show a day	10.0	62.1
Show a week	0.0	3.4
Show a month	0.0	10.3
Number of days in week	0.0	27.6
Counted days	0.0	10.3
Number of weeks in month	0.0	17.2
Counted weeks	0.0	10.3
Number of months in year	0.0	31.0
Show on calendar	0.0	10.3
Name of this month	0.0	34.5
Knew birthday month	52.2	82.8
Named other months	0.0	75.9
Named months in order	0.0	3.4
SEASONS		
Current Season	26.1	69.0
Named and described		
Winter	52.2	62.0
Spring	4.3	34.5
Summer	34.8	41.4
Fall	0.0	24.1
SPECIFIC TIMES		
Named clock or watch	69.6	93.1
What time (clock face used)?		
8:00	4.3	27.6
12:00	4.3	51.7
6:00	4.3	20.7
4.30	0.0	3.4

Responses to questions about liquid measure, avoirdupois weight, and length are presented in Table 2.

Both pre-school groups seemed to have little familiarity with the conventional liquid measures. Except for the quart milk carton,

TABLE 2

PERCENTAGES OF CORRECT RESPONSES ON QUESTIONS ABOUT LIQUID MEASURE, AVOIRDUPOIS WEIGHT AND LENGTH

	N.S. *N*=23	Kgn. *N*=29
LIQUID MEASURE		
Quart milk Carton	47.8	96.6
Quart	8.7	24.1
Pint Carton	0.0	13.8
Halfpint Carton	0.0	0.0
Quart jar	0.0	13.8
Pint jar	0.0	17.2
Number of cups in pint	0.0	24.1
Gallon		
Give me some gas	13.0	20.7
Give me _____ gallons	8.7	51.7
Fill 'er up	0.0	20.7
Give me $_____ worth	8.7	0.0
AVOIRDUPOSE WEIGHT		
Pound	13.0	20.7
How Much Meat		
Weigh	0.0	24.1
Scales	13.0	31.0
Personal weight	17.4	13.8
to find one's weight		
Pounds	17.4	17.2
Scales	39.1	75.9
LENGTH		
Named ruler	0.0	34.5
Named yardstick	13.0	10.3
Uses of rulers and yardsticks	57.9	62.1
Show an inch	8.7	41.4
Yardstick (36 inches)	4.3	10.3
Counted	0.0	3.4
Foot ruler (12 inches)	0.0	27.6
Counted	0.0	17.2
Use of speedometer	8.7	72.4
Used miles per hour	4.3	44.8

TABLE 3

PERCENTAGES OF CORRECT RESPONSES TO QUESTIONS ABOUT MONEY AND GROUPS

	N.S. N=23	Kgn. N=29
MONEY		
What is it?	78.3	96.6
Use of money (to buy things)	78.3	100.0
Identification of		
Penny	82.6	96.6
Nickel	30.4	75.9
Dime	26.1	79.3
Quarter	4.3	34.5
Half dollar	0.0	31.0
One dollar bill	52.2	79.3
Relative worth		
Which will buy more?		
Nickel and penny	17.4	0.0
Nickel and dime	17.4	41.4
Values of		
Nickel—five pennies	4.3	27.6
Dime—ten pennies	4.3	20.7
Dime—two nickels	0.0	24.1
GROUPS		
How does your mother		
buy eggs?	0.0	20.7
Dozen	0.0	27.6
Number in Dozen	4.3	41.4
What are two shoes called that		
go together? Pair	4.3	79.3
Number in pair	52.2	37.9
Described pair	4.3	44.8

which was recognized by almost all the kindergarten children and almost half in the nursery school, other common measures were recognized by only a few of the children. "Gallon" was known by many of the children in the context by buying gasoline, the test question. Whether or not they knew this measure in other contexts was not tested.

Children of both age groups performed about the same on questions regarding avoirdupois weight. Neither group demonstrated much understanding in these questions. By kindergarten, some children were using the verb "weigh" instead of the noun "scales" to answer "How does the man in the store knew how much meat to give your mother?" As indicated by this test, concepts of weight are just forming during the preschool years.

More than half of both groups knew the uses of rulers and yardsticks but the children were relatively unfamiliar with the other questions about length. Use of a speedometer was known by more kindergarteners than nursery school children. More older children also used the term "miles per hour" than did the younger children.

More older children knew the term "dozen" than did the pupils in nursery school. More kindergarten children knew "pair" and its description than did the younger children, although many of the younger children knew the number in a pair. The children apparently were more familiar with *bair* than they were with

Table 4

Percentages of Correct Responses to Questions about Miscellaneous Measures

	N.S. N=23	Kgn. N=29
Identify wall thermometer	26.1	65.5
What does it tell	21.7	62.1
How do you know how hot or how cold it is?	17.4	72.4
Where is the red line when it is hot?	17.4	51.7
When it is cold?	21.7	48.3
What do we say (degrees)?	8.7	20.7
Identify clinical thermometer	13.0	48.3
What does it tell?	26.1	69.0
How is your house heated?	00.0	79.3
Coal (tons)	00.0	00.0
Gas (meter)	00.0	3.4
How do you make your house light when it is dark outside?	56.5	89.7
What makes the lights (electricity)?	13.0	75.9
Identification of light bulb	34.8	86.2
What are they for?	56.5	82.8

dozen. Percentages of correct responses to questions about groups and money are presented in Table 3.

Money was familiar to all the children. Pennies were the best known coins to both groups and a dollar bill was known by many in both groups. More kindergarten children than those in nursery school knew the names of a nickel, dime, quarter, and half dollar and both groups knew these coins in this relative order. The relative value of the various coins was little known by both groups.

Percentages of correct responses to questions about other measures are given in Table 4. Kindergarten children were better able than the nursery school children to identify wall and clinical thermometers and to explain their use. However, the groups did not differ greatly with regard to ability to explain specifically the relation of temperature and position of "the red line" on the thermometer. The older children also understood better than the younger children how their houses were heated and lighted. One kindergarten child explained that electricity came through the wire and caused the light to burn.

Conclusions

As indicated by expressions of children enrolled in a nursery school and kindergarten, marked differences in familiarity with measurement were noted between responses of four- and five-year-old children. While differences between the groups were found for most items, the older children had not achieved mastery of the ideas. This is consistent with previous research evidence about concept development that indicates that growth, while substantial between age groups, is continuous and, for different children, occurs at varying rates.

With most of the measures, some of the four-year-olds interviewed had at least a slight familiarity. Some of the older children, likewise, had only little understanding of particular measures. For some children in both groups, familiarity with various measures was extensive and accurate.

To generalize from these data is dangerous. However, some conclusions may be expressed with caution. Pre-school children have some understanding of common measures. Significant growth may occur between the nursery school and kindergarten years for some common measures. These findings lend encouragement to the belief that pre-school-age children might profit from direct experiences designed to foster familiarity with common measures and measurement. If subsequent research confirms hypotheses based on this belief, implications abound for arithmetic instruction in the early elementary school. Certainly these present findings indicate that during the pre-school years of four and five, children have the beginnings of measurement concepts and that these concepts may develop.

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EDITOR'S NOTE. This study indicates that nursery school and Kindergarten pupils are in the formative stages of concept formation. The authors suggest that at this level the curriculum might well include more planned work in the development of concepts of measurement. There is a long progression from the early recognition of a term and the acquaintance with its use in some context to a satisfactory concept and knowledge of the use of the measure. For example, the *pound* may be spoken rather glibly in several contexts and yet a satisfactory concept of pound may be far removed. Likewise, a quart may initially be represented by a carton having a distinctive shape without any real understanding of the aspect of volume. Much later the "volume-use" is affected by the association with a specific content as for example buttermilk or ice cream. And at some stage the distinction between liquid quart and dry quart must be established.

The editor was pleased to note the early familiarity with measurements of time even though these were extremely simple and crude. In his own investigations he had found intermediate grade children had not refined their concepts and uses of the measurement of time as much as desired. It is our job in the schools to foster not only the concepts of measures but also to develop a rich association with uses of these measures.

Editor's Note—What Numbers Mean to Young Children

EDITOR'S NOTE. The Gundersons have compiled an excellent list of the number concepts held by seven-year-old pupils in one school. The lists do not indicate how many of these concepts were given by more than one pupil. It is fair to assume that pupils within one school that works at the job of developing ideas in arithmetic have acquired considerable common knowledge. The recent study shows a great deal more learning than the previous study. No doubt part of this is due to the increased emphasis upon arithmetic and the growing consciousness of teachers in kindergarten and grade one that pupils of this age not only can but want to learn about numbers and quantity. It is probably unfair to infer that the Long Island community is more sophisticated than Wyoming. This study shows what is being done in one school and should be full of suggestions to those teachers who have viewed arithmetic as skills to be memorized. It seems very probable that the next five years will show an increase in the arithmetic program in kindergarten and grades one and two. Fortunately many people are dropping the unfortunate view of arithmetic readiness that was common but a few years ago. Let us move ahead.

Some Observations on the Contemporary Mathematical Scene*

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TEACHERS OF MATHEMATICS find that they are in a dilemma at the present time that is both interesting and troublesome. From one horn of the dilemma flies the banner signifying great popularity, for mathematics teachers have suddenly become indispensable; someway they have become tied up with national survival. From the other horn flies the banner signifying gross unpopularity, for mathematics teachers, according to many emphatic reports, are poor teachers, and they have been teaching the wrong things.

I can testify personally in regard to the drastic and favorable change that has taken place in the popularity of mathematics and of mathematics teachers. In fact, I had great trouble obtaining a position when I first appeared upon the world scene with my credentials in mathematics. Many times, when applying for a job, I was told by school administrators that it was unfortunate indeed that I was unprepared to teach some useful subject; if I could only coach football. In the end I did coach girls' basketball; I taught chemistry, general science, and public speaking, coached all the school plays, and did substitute teaching in home economics, thereby teaching seven classes a day, to make up a package deal that permitted me to teach a few classes in mathematics. My salary for this back-breaking assignment was $1,500, raised two years later to $1,600. Thirty-five years ago I found it necessary many times to restrain my active temper when, it seemed to me, every educational conference brought forth such statements as "Mathematics, along with the classical languages, is a dead subject"; "Sound training in good citizenship is the basic purpose of the high school"; "Mathematics is good only for its disciplinary value"; "Some mathematics should be taught as a background for those who may study engineering"; "No mathematics should be required; let it be chosen by the very few who need it."

I can testify also in regard to the deserved unpopularity of some mathematics teachers and of the subject they teach, for I have children. The interest and accomplishment in mathematics of each of my children has varied from the winning of prizes and scholarships to the recording of marks one rung above failure. I have watched my children flounder in the mire as some teacher inspired by a super modernist spirit tried to teach the most sophisticated kind of mathematics without any attention to the necessary groundwork and without the slightest comprehension of the maturation process; I have seen their interest lag as some teacher, an undoubted relic of the days when mathematics, like castor oil, was regarded as one of the necessary evils of life, tried to peddle his wares; and I have observed their enthusiastic accomplishment when a teacher of personal scholarship and of wise understanding stimulated them to seek mathematical knowledge. Certainly we must continue our struggle for more and better teachers upon every academic level, especially, I think, in our colleges and universities.

* This is a summary of President Newsom's address to the New Jersey Association of Mathematics Teachers, May 2, 1959.

Curricular Problems

Although I believe the situation may have been exaggerated, I am quite convinced that we also have some curricular problems that require attention. There is little doubt that pre-college mathematics now gives great attention to some ideas of little significance; likewise some concepts of modern importance, now ignored, deserve consideration. It appears to be essential that students entering college during this period of rapid technological advance be prepared to register for courses in the Calculus in order that they may cope at the earliest possible moment with many modern concepts of fundamental significance. In general, it seems evident to me that the curriculum in mathematics that precedes the Calculus needs considerable reorganization, and I suspect that penetrating attention should be given to matters of emphasis. Deliberations upon such suggestions, however, must give due recognition to many factors; crash programs for junking the old and installing the new frighten me. I was badly shaken recently when an outstanding high school teacher said with some emotion, "I wish I knew what to teach." Such a state of mind is unnecessary. Let's keep doing our best job within our available resources, but with our eyes and ears on the alert for opportunities to participate in programs that are better. This is not a time for hysteria; it should be a period of fact finding, of evaluation, of conscious concern for improvement.

Mathematics and Arithmetic

I hope I will not muddy the waters that are already murky if I use the remainder of my time in making certain observations that are the result of long hours of study of the educational scene. I believe that I have had the advantage of making this study from many vantage points.

First, and I believe that this may have some significance for such a group as this, I am concerned by modern efforts to mathematicize the teaching of arithmetic. Let me explain what I mean. Little can be done in furthering a student's intellectual growth unless he has attained a high degree of perfection in the utilization of certain tools of the mind, among them being arithmetic. In addition, of course, arithmetic is essential in carrying out daily household chores, in commercial transactions, in accounting, and in finance; facility in its use is virtually indispensable in a civilized society. It would be the rare mathematician, however, who would understand or be competent in arithmetical skills. Admittedly all operations with number are premised upon certain basic propositions of the appropriate algebra, but skill in the use of numbers requires a specialized kind of experience. Yet, educators and the public continue to confuse mathematical ability and arithmetical ability, mathematical activities and arithmetical activities, to the detriment of boys and girls in the grades. Specifically, therefore, I want arithmetic teachers to be free of mathematics teachers in the development of their subject and in the way in which they present it.

The time must come, of course, when some day, perhaps in the seventh or eighth grade, the announcement is made that "we shall now start the study of mathematics." This might be followed, in the first instance, by the introduction of the set of propositions that characterize the integers, indulging in appropriate inductive studies as a prelude. As the class progresses in its comprehension, the nature of mathematical systems could be discussed in simple terms, and students would then have their first experiences with deduction,—thereby leading to a systematic development of the principles of traditional algebra. Of course, the extent to which the teacher ultimately goes into various other mathematical systems will depend upon the ability and interest of the class as well as those of the teacher.

In presenting such material as I have just suggested the good teacher will always remember that rigor is a relative term; this is fundamental. As Felix Klein so often pointed out, that which is best from a pedagogical

point of view is not always the most logical by the teacher's standards. Even though mathematics, by definition, is a deductive science, its presentation to students and its natural development will involve a strong inductive element. This is true in the systematic study of geometry as well as in the development of the number concept. Mathematicians are too sensitive on this score; an author of some important research, I have noted through the years, will publish an elegant deductive demonstration of his results, but he feels it is unethical to reveal the long and laborious process, usually inductive, that has preceded the ultimate deductive organization. For too long we have made the mistake of plunging the student of plane geometry into a discipline highly organized along deductive lines. Of course, the deductive should be shown and studied but that type of organization, it must be emphasized, reveals the subject matter of the course in its most advanced stage of development. By contrast, most elementary courses in algebra, which are really courses for the systematic development of the properties of number, sin in the reverse direction by never showing any part of the ultimate logical organization which mathematicians find to be necessary.

Every teacher must keep in mind that mathematics in a technical sense is a triumph of applied logic, but the devices of logic are not simple, they are not inherited, and the teacher of mathematics must take each student by the hand as he leads him through the complexities that are involved. As teachers, let us not forget that genuine mathematical understanding cannot precede the student's comprehension of the logical and his ability to read and understand.

A good teacher cannot afford to ignore the historical background of his subject. The biologists have their theory of recapitulation which states that to a great extent each individual in his early development reproduces in himself the early evolution of the race. I sincerely believe, more than we sometimes realize, that each student develops in his understanding of mathematics along much the same line that history reveals that man developed. This is especially true, I believe, of our students of arithmetic. It is apparent, of course, that too rigid an interpretation can be placed upon this statement, but I, as a teacher, must admit my own indebtedness to my study of man's historical struggle to build the remarkable system called mathematics. A knowledge of history is imperative for other reasons. The evolution of such a discipline as mathematics can not be separated from the rest of the stream of history; mathematics has been a contributor and a recipient in terms of the other factors of civilization. The fact that mathematics is a living body should dominate the attitudes of the teacher and should be portrayed to our students.

Mission of the Teacher

In the classroom and out I believe sincerely that mathematics teachers have a particular mission during these times that, properly carried out, can pay tremendous dividends to our profession and to the society of which we are a part. In truth, there has been no better period in history for initiating the crusade which I propose. Specifically, the time has come to make certain that mathematics receives proper definition in the minds of the so-called educated man. Let it be taught that mathematics reveals human wisdom at its best, that the symbolic models of mathematics—truly human creations—must be classified along with the finest of literature, art, and philosophy when describing the humanities; in fact, mathematics is one of the humanities, not one of the sciences. It is true, of course, that modern science, especially physical science, has had great success in studying relationships within nature by using mathematical models as coordinating mechanisms; this fact enhances the prestige of mathematics, and that of its creators, but it has no great relevance when considering the actual nature of mathematics. One may predict with

considerable certainty that mathematical models will soon be used extensively in the social sciences; in truth, we know of no alternative to the doctrine that their utilization must be fundamental in all of man's efforts to comprehend and control his environment. Moreover, and this is most important, we shall find most certainly that studies of possible inherent limitations upon the human mind as an interpreter of the structural properties of nature will be concerned with many aspects of the field of knowledge presently considered under the title, "Foundations of Mathematics"; that is, one who seeks to understand the actual meaning and nature of human knowledge, other than that which is purely descriptive, will be obligated to study mathematics, its concepts and its methods.

Suggested Modifications

If one accepts my beliefs on such matters, it is apparent that certain common practices need modification. First, although I believe sincerely that most mathematicians can profit from varied experiences that involve mathematical application, it does not follow that a knowledge of some kind of science is a substitute for a knowledge of mathematics, or vice-versa. When a school or college specifies electives by groups, as part of the prescribed graduation requirement, mathematics can be listed properly only among the humanities. Second, the study of mathematics, in the light of my comments above, is one of the two or three basic elements in any school or college program designed to provide a general education. Those who disagree, I find, were raised on a diet of useless algebraic and numerical manipulations, so they are unfamiliar with the actual nature of mathematics, its uses and its values. Let us not duplicate such an unfortunate educational experience for future generations of citizens, including perhaps some school administrators, who in turn may be called upon to pass judgment upon the significance of mathematics.

Certainly mathematics teachers now have a challenge and a responsibility that is unprecedented in educational history. I am glad that I have lived to be a part of this day. What a wonderful experience it is to be, through general acknowledgment, an essential part of the greatest technological revolution within history, of the trying but exciting world revolution! What an opportunity we have been given to show our students at least a glimpse of their true capacities as men and women!

EDITOR'S NOTE. President Newsom indicates that a deductive presentation is usually preceded by a long inductive study. He also indicates some faith in the doctrine of recapitulation. If we put these two together we have a keynote to teaching and learning. We might ask the question, "Is it genuine learning when a pupil memorizes the conclusions already reached by another instead of recreating at least some of the steps in establishing the conclusion?" For example, in "learning" the number combinations of addition and multiplication, should a pupil memorize the answers or should he establish, through some mode of discovery, the answers and then proceed to master them so that he can use them? In a proof in geometry, should a pupil seek a proof himself or should he reproduce a proof that someone has printed in a book? Which procedure will give him greater insight and perhaps develop in him qualities that he can and will use after he has turned the page in the book? In seeking answers to these questions we might find that "arithmetic" and "mathematics" which Dr. Newsom has separated have much in common. Let us not agree that arithmetic is only a set of skills to be mastered for use in contemporary society and in future courses in mathematics.

Discovering the Multiplication Facts

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A PUPIL MAY LEARN the basic facts in multiplication by rote or he may discover them for himself. Usually a pupil finds it advantageous to have objective aids of some kind to help him find the answers to different number groupings in this process. A number grouping consists of any combination of two one-place numbers.

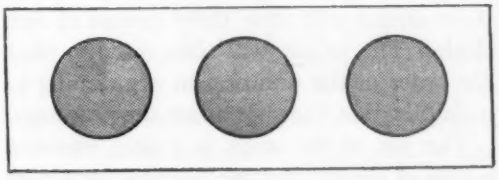
There are three stages of discoveries which may be identified in learning the basic facts in multiplication. First, the pupil should discover a fact through the use of objective or visual aids. Second, he should derive a fact from other known facts. Finally, he should discover a pattern that characterizes the series of the products of a given table.

Many different kinds of manipulative materials may be used to have the pupil discover a multiplication fact. Markers, such as disks, beads, or splints are used frequently. Often pupils drop these materials on the floor so as to create a disturbance in class operation. For that reason many teachers use such manipulative materials sparingly.

Strips Containing Geometric Designs

Markers made of construction paper are effective aids in helping the pupil discover the answer to a grouping in multiplication, such as 3×5. Each pupil should have a set of nine or ten strips of construction paper approximately one inch wide. The length of the strip would depend upon the facts to be developed.

If the threes in multiplication are to be developed, each strip should contain three designs, such as circles, squares, triangles, or

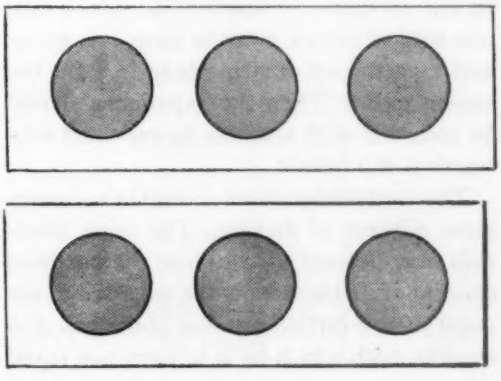


diamonds as shown. The teacher should plan to have at least three different kinds of designs distributed among the pupils. All of the designs for any pupil should be the same. The designs may be grouped in rows on one sheet of paper. Then the paper should be folded between consecutive rows so as to form strips.

Suppose the teacher plans to introduce the facts involving threes. Each pupil would use his materials to show the answer to a grouping, as

$$\begin{array}{r} 3 \\ \times 2. \\ \hline \end{array}$$

The teacher would have the pupil show two groups of 3 circles, 3 squares, or 3 triangles. A pupil having markers showing three circles would read the fact as, "Two groups of 3 circles are 6 circles." Pupils having other geometric patterns would read the



fact in a manner corresponding to the designs on the strips. Then the class could generalize about two groups of 3's. The fact should be read, "Two groups of 3's are 6." The word "times" should not be used in initial instruction of the multiplication facts.

By turning the strips through a 90 degree angle, a different grouping is formed. Now the grouping which showed two groups of three circles will show three groups of two circles. The pupils will soon discover that the order of the numbers in a grouping in multiplication does not affect the product.

The use of the strips is a very effective means of developing the division facts with the multiplication facts. Suppose a pupil shows 6 circles in strips of 3 circles each. The teacher would question the pupil as follows:

How many circles do you have in all?
How many circles are in one group?
How many groups of 3 circles are in 6 circles?

$$\begin{array}{r} 2 \\ \hline 3 \text{ circles})\overline{6 \text{ circles}} \end{array}$$

$$\begin{array}{r} 2 \\ \hline 3)\overline{6} \end{array}$$

Then the teacher should write the fact as shown, first with circles, then with squares or triangles, and finally without any label. The teacher should question the pupil to see if he understands that the 2 in the quotient represents the number of groups. Also, the teacher should have the pupil demonstrate how he could find the number of groups if he did not know the answer. He should take one strip of circles away to form one group and take the other strip away to form the second group. Then the experience should be recorded with symbols by repeated subtraction of 3 from 6.

The method described shows the measurement concept of division. The same materials may be used to introduce the partition concept of division. Now the teacher has the pupil show a certain number of designs, as 6 squares, with which he is to form two equal groups of squares. In arranging the groups the pupil finds that there will be 3 squares in each group. This is the same as finding half the number of squares.

The language usage in teaching the basic facts in multiplication and division is very important. The pupil must understand that he is dealing with groups. The four fundamental processes are ways of dealing with groups. Such expressions as "times," "multiplied by," "into," and "divided by" do not suggest the meaning of groups. In multiplication the pupil should understand that the process involves the forming of one group from two or more equal groups. Thus, if the pupil has 3 groups of 4 cents each, he has a total of 12 cents.

In division the pupil must decide if he is to find the number of equal groups formed when the number in one group is given, or if he is to find the number in each group when the number of groups is given. It may be necessary to find the number of groups of 3 cents each that can be formed with 12 cents, or to find the number of cents in each of three equal groups.

Relationships among Facts

The first stage in the discovery of the multiplication facts consists in using strips of paper containing grouped geometric designs. The second stage consists in discovering relationships among known facts to enable the pupil to derive a new fact. At this level of development, objective and visual materials are not used.

The teacher should have the pupils construct a table of the multiplication facts, such as the table of the 3's. The pupil knows all of the basic facts of the 3's, but he should discover each succeeding fact in a table from his knowledge of relationships of certain given facts. Suppose the known elements of the table of 3's are those shown below.

$1 \times 3 = 3$
$2 \times 3 = 6$
$3 \times 3 =$
.....
$10 \times 3 = 30$

From these known facts the pupil should discover the remaining facts in the table. Each new fact he has discovered may be used in finding the other facts.

To find the first missing fact in the table, the pupil may discover that since one 3 and two 3's are known, three 3's must be the sum of 3 and 6, or 9. To find four 3's, he could double the product of two 3's, or he could add one 3 to the product of three 3's. To find five 3's, he could add the products of two 3's and three 3's, add one 3 to the product of four 3's, or he could find half the product of ten 3's. Similarly, the pupil should discover different ways of finding each unknown fact in the table. The method described may be used in constructing any of the tables in multiplication. Naturally, slow learners will make fewer discoveries than those pupils who develop insight into number.

Discovering Patterns of the Product

The third stage in the development of the multiplication facts consists in discovering a pattern which characterizes the series of the products of a given table. The tables may be grouped into families. The first family consists of the *twos* which includes the table of 2's, 4's, and 8's. The next family consists of the *threes* which includes the table of 3's, 6's, and 9's. The two remaining tables are individual families of the table of 5's and the table of 7's.

The multiplication products follow in an orderly sequence. The superior pupil should discover this sequence in order to enrich his understanding of the structure of the number system. Much of modern mathematics originated in and deals with number series. The more capable pupils in arithmetic may be challenged to discover something about the sequence of the products in a given table. Discovery of patterns in a series of products should be used to enrich the program of the superior pupils.

The column on the right of the table of 3's shows the series or sequence formed from the products. The first three products are one-place numbers. The remaining products are two-place numbers. The sum of the digits in each of the two-place products forms the repeating series 3-6-9. The lines designate different decades.

1×3= 3	3
2×3= 6	6
3×3= 9	9
4×3=12	3
5×3=15	6
6×3=18	9
7×3=21	3
8×3=24	6
9×3=27	9

The column on the right of the products in the table of 6's shows that the repeating series of the sum of the digits in the products is 6-3-9. Since the sum of the digits in 48 is a two-place number, find the sum of the digits in 12. This sum is 3. Then the series is complete.

The pupil may discover that the digits in ones' place in the products follows the sequence 6-2-8-4-0 and then repeat. This series uses the same numbers found in the series for the family of the *twos* as shown later.

Most teachers are familiar with the pattern of the products in the table of the 9's. The sum of the digits in each product is 9.

1×6= 6	6
2×6=12	3
3×6=18	9
4×6=24	6
5×6=30	3
6×6=36	9
7×6=42	6
8×6=48 (12)	3
9×6=54	9

From an examination of the products given, it is seen that

9 18 27 36 45 54 63 72 81

the digit in ones' place decreases by one in each succeeding product while the digit in tens' place increases by one. This is true because adding 9 is the same as adding 10 less 1.

The digits in ones' place of the product of the table of the 2's follow the sequence 2-4-6-8-0 and then repeat. For the table of the 8's, the sequence is formed in the reverse order, beginning with 8 and ending with 0. This is true because 8 is two less than the decimal base of our number system. All of the symbols are repeated in sequence as numbers progress in value from 10 or a multiple of 10. For the 4's the digits in ones' place follow the sequence 4-8-2-6-0 and then repeat. These digits are the alternate numbers in the series of the table of the 2's.

Most pupils who learn the tables of the 5's discover the sequence of the endings of the products. The sequence of the digits is 5-0 with repetitions. In the table of the 5's, each product of odd numbers ends in 5 and each product of even numbers ends in 0. Since 5 is half of the number base, the product of 5 and a number must end either in 5 or 0.

There is no apparent pattern for the 7's because the table of 7's is not in a family of any of the other tables and the number 7 is prime with respect to the number base. The digits in the ones' place of the products of the grouping of the 7's in multiplication are as follows: 7-4-1-8-5-2-9-6-3. If the table is extended to include the zero in ones' place in the product, 10×7, the series will include each of the ten digits. This is true only in the tables of 3's, 7's, and 9's.

From the above series it is readily seen that there is a difference of 3 between successive numbers providing the number preceding is greater than the number following. When the number preceding is smaller than the number following, as 1-8, the 1 must be considered 10 more, or 11. Since 7 is 3 less than the base of our number system, each succeeding product will be increased by 10 less 3.

Beginning with 7, each succeeding product will be in the next decade except at the two points marked in the series shown.

7 4 1–8 5 2 2–9 6 3

The digits 1-8 are the endings of the products of the 7's in the twenties and 2-9 are the endings of the products in the forties. In each of these cases a smaller number precedes a larger number in the series.

Many of the discoveries of the patterns of the series are intended for the superior puils at the upper level of the elementary school. The teacher must bear in mind that this phase of the work in multiplication is for purposes of enriching the program of the pupils who develop a keen insight into the structure of the number system.

Summary

This article is intended to implement the principle of discovery in the teaching of the multiplication facts. Three different stages of discovery have been identified. First, the pupil should use such materials as rectangular strips showing groupings of geometric designs, as squares or circles, to discover the multiplication facts. Second, without the use of objective or visual aids, he should discover relationships among known facts so as to form a table of the facts. The number of discoveries a pupil will make depends upon his insight into number. Finally, the pupil who excels in number should be encouraged to discover a pattern in the sequence of the products. The tables should be grouped into families to enable a pupil to detect a pattern descriptive of the products of a given family.

The plan described establishes a minimum program for the teacher to follow in introducing the multiplication facts. The use of manipulative aids should enable a pupil to understand the meaning of a multiplication fact. The application of the other two procedures described should help the teacher to differentiate the program for the superior

[*Concluded on page 208*]

A Comparison of Test Results

North Carolina, California and England

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"THE ARITHMETIC TEACHER" for February, 1958, carried an article on comparison of arithmetic achievement in English and California schools by G. T. Buswell.[1] The reported results were viewed with alarm by some and raised a number of questions regarding arithmetic achievement in the United States. Some of these questions were related to the training of teachers who are to teach arithmetic and others were related to achievement of students at the point of terminal training. It was decided that a study, centered in selected North Carolina schools, might provide some clues regarding the answers to these questions.

One question raised by Buswell could be easily settled by using the North Carolina schools for white children. The possible effects of a fairly large proportion of Negro and Spanish American children in the California sample were eliminated simply by selecting white schools in North Carolina whose populations contain no Negro students and a very limited number of Spanish American students.

A second point raised by Buswell's comparison of age groups, 10 years, 8 months, to 11 years, 7 months, in England and in the United States was whether terminal training in England and terminal training in the United States produced comparable results in arithmetic achievement. Arithmetic training in England is normally completed at the end of the sixth year of elementary education. Terminal training in arithmetic in the United States is normally completed at the end of the eighth grade or the eighth year of elementary training. It was decided to make a comparison of eighth grade achievement in selected North Carolina schools with sixth year achievement in the English groups based on the examination used by Buswell.

A number of similar explorations were made in the arithmetic achievement, as measured by this particular examination, of college students. This investigation was particularly concerned with college students preparing to teach arithmetic in the elementary school on an elementary credential or in the high school on the secondary mathematics credential. These results will be reported in another article.

The characteristics of the testing device described by Buswell and used in the study are as follows: The test, containing 100 items, was originally patterned after the "11 plus examination" given to English students. An adaptation was made by Buswell to assure the use of 100 items, but at the same time to eliminate those items employing peculiarly English arithmetic skills, such as money conversion. The result of this adaptation left 70 test items on which the California comparison was made and on which comparison was made for this study. The test itself was divided into two sections, one section of 40 numerical items and a second section of 60 word or descriptive problems.

The initial problem of this study, limited in funds, was the selection of schools in which the tests might be administered. The original tests were given to groups of urban and rural students, and the results indicated that the urban students scored slightly higher. Presuming that this differential

[1] G. T. Buswell. "A Comparison of Achievement in Arithmetic in England and Central California." THE ARITHMETIC TEACHER, 5: 1–9, February, 1958.

would not change, the North Carolina study was limited to urban schools. It was decided that schools of small or medium-sized cities would be used in order that an entire grade group could be examined. These cities selected would, in the judgment of the research group, be representative of the North Carolina urban school population and would include a cross section of the North Carolina urban school population.

With these ideas in mind, and with the realization that the sample would be necessarily small, it was felt that results would be significant, at least for North Carolina uban schools.

Distributions of scores for North Carolina students showed some appreciable differences between North Carolina groups and groups previously tested (Table I). The first point of difference lies in the fact that the North Carolina 10-8 to 11-7 age group scored significantly higher than the California urban group although still significantly lower than the English urban group. Since the North Carolina group was taken from all white schools, it is probable that the removal of the Negro and Spanish American groups is a factor in the differences between North Carolina and California school groups. There may, of course, be other differences. However, teacher training levels are comparable between North Carolina and California as are average class sizes.[2,3] Other differences that may affect the situation are relative stability in school population in North Carolina schools and possibly a higher degree of uniformity

[2] *Enrollment, Teachers, and Schoolhousing.* (Washington: U. S. Department of Health, Education, and Welfare, Circular No. 551, 1958), p. 12.

[3] *Rankings of the States.* (Washington: National Education Association, 1957), p. 10.

TABLE I

DISTRIBUTION OF SCORES FOR GROUPS TESTED SHOWING MEAN AND 10–90 PERCENTILE RANGE

Scores	English Total[a]	California Total[a]	English Urban[a]	California Urban[a]	N. C. 10-8 to 11-7	N. C. 8th
69–70	7		7			
66–68	43		40			1
63–65	62		57			1
60–62	74		73			1
57–59	111		95			3
54–56	131		117			7
51–53	111	1	99	1	1	19
48–50	128	2	114	2	0	17
45–47	148	4	126	2	0	29
42–44	138	1	110	0	2	23
39–41	124	5	106	5	2	40
36–38	122	7	97	6	6	42
33–35	149	16	116	13	7	63
30–32	131	28	115	27	9	50
27–29	128	45	106	34	14	50
24–26	165	66	142	56	24	51
21–23	147	126	127	111	23	50
18–20	158	249	135	205	22	32
15–17	175	389	152	328	25	31
12–14	193	555	161	443	28	30
9–11	202	719	170	559	12	11
6– 8	230	523	197	384	14	4
3– 5	216	317	171	218	4	
0– 2	98	126	70	89	1	
N	3191	3179	2703	2483	194	555

----Mean.

I 10–90 Percentile Range.

[a] G. T. Buswell. "A Comparison of Achievement in Arithmetic in England and Central California," THE ARITHMETIC TEACHER, 5: 3, February, 1958.

resulting from state adoption of texts and supplementary materials.

The comparison between the groups at the point of terminal training in arithmetic, that is the English urban group, 10 years, 8 months, to 11 years, 7 months, with the North Carolina urban group of eighth graders, to whom the test was administered in January, 1959, or approximately the midpoint of the eighth grades, indicates that there is no significant difference in levels of achievement in the two groups as measured by the total test.

However, there are a number of differences that exist between these groups which may be due to certain factors within the testing device and involving the maturity of the students. It may be noted that the distribution of scores for the North Carolina eighth grade group more nearly approaches a normal distribution, being less positively skewed than the English distribution, indicating that the test is probably a slightly better instrument in its application to the North Carolina eighth grade group than it was in its original application to the English group. It is possible that maturity is the factor that provides for a more nearly normal pattern for the eighth grade group. A second difference lies in the actual significant differences between group means on the two sections of the test (Table II). The North Carolina eighth grade group is significantly below the English group on the first section, which involves a considerable number of problems in which denominate numbers and conversions are employed. On the second section, the reverse is true. In these problems, which are primarily word problems involving reading ability and concept mastery, the North Carolina eighth grade group did significantly better than the English group. It might also be noted that the North Carolina group, 10 years, 8 months, to 11 years, 7 months, did nearly as well as the English group on the second section although they were much lower on the first section of the test.

Several points might be made in relation to this study. One, already pointed out by Buswell, lies in the differences in instruction between the English and American groups. Where the daily life in England requires rapid and accurate conversions in the various areas of measurement and particularly in money, this requirement is not emphasized so much in America. It is expected that the American student master the skills necessary to make these conversions, but speed is not a prime factor. As a result, the pattern of achievement on the first section of the examination indicated that few American students in any of the groups had time to finish, even though a comparatively high proportion of the problems actually finished were done correctly. The emphasis in American schools on concept, word problems, and interpretation of arithmetic situations is apparent from the much higher range of scores on the second section of the examination.

There is the undeniable fact that, as measured by this particular test, American schools are taking two extra years in which to assure achievement levels equivalent to those attained in the English schools. This conclusion is based on the substantially equal achievements of the English sixth year group with the North Carolina eighth grade group. There is, however, a possible fallacy. The conclusion that the examination used in the experiment actually measures achieve-

Table II

Means and Standard Deviations by Groups on Section I, Section II and the Total Test

	English Urban 10-8 to 11-7[a]	California Urban 10-8 to 11-7[a]	N. C. Urban 10-8 to 11-7	N. C. Urban 8th
Section I				
Mean	14.3	4.3[b]	6.2[b]	9.2[b]
S.D.	8.8	2.1	2.8	4.5
Section II				
Mean	15.3	8.2[b]	13.8[b]	21.7[b]
S.D.	10.6	5.2	6.8	8.1
Total				
Mean	29.6	12.5[b]	20.0[b]	30.9
S.D.	18.8	6.8	8.9	11.5

[a] G. T. Buswell. "A Comparison of Achievement in Arithmetic in England and Central California," The Arithmetic Teacher, 5: 3, February, 1958

[b] Differences from English means significant at 1% level.

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ment in those skills and understandings that we wish to teach in American schools may not be correct. The test, after all, is English in origin and the problems are oriented to needs of the English society.

To pursue the comparisons to logical conclusions would imply that we devise two different types of examinations. First, an arithmetic mastery examination in those skills that are commonly sought for both English and American terminal arithmetic groups and second, an examination which employs those skills expected of the American students at the point of terminal arithmetic training. In the first case, we would get the comparison on a basis that does not involve a large proportion of cultural differences. In the second case, the investigation, possibly using some existing standardized arithmetic tests, would measure American and English achievement on an examination culturally oriented to America or to the needs of students in American society. This second test, taken in conjunction with the examination used in this particular piece of research, would probably give a complete view of relative achievement levels. No single examination, however, unless specifically devised to test equated groups by use of culture-free items, can be expected to give accurate results and permit valid comparisons.

EDITOR'S NOTE. It is worthwhile to compare the learning of different groups when we do so from the point of view of asking honestly whether we are achieving in scope and in quality and quantity as much as we should reasonably expect to achieve. Dr. Tracy points out some of the factors that make the comparison of English results with those in this country difficult to assess properly. Professor Buswell's original article has served well to stimulate school people to question their programs, procedures, and results. Let us continue this questioning.

Dr. Tracy suggests that someone prepare a test that will feature the common elements of instruction and mirror the common culture of this country and England. This would be a good project for someone who holds a Fulbright or other foreign travel fellowship. For further information on comparisons of our schools with those of other countries the reader is referred to "A Comparison of Achievement in Arithmetic in England, California, and St. Paul" by Thomas L. Bogut in THE ARITHMETIC TEACHER, March, 1959 and "Glen Rock Scores on a Canadian Test" in the issue of October, 1958.

Experiments in the Primary Grades

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DO THE NEW IDEAS REALLY HELP the children learn, or are they just more dramatic? If a teacher tries to teach arithmetic using some of the new materials, whether or not she continues to use them, her thinking about her own methods has been stimulated. She can evaluate her teaching in a new light. She can judge her teaching in a more scientific way, having experimented, observed, and drawn her own conclusions.

In the first and second grades, children are often exposed to the abstract too soon. They may learn through visualization and by manipulation the concepts of the numbers from one to ten, but then they are suddenly required to start learning the basic addition and subtraction facts with little help from concrete objects. The facts are learned more or less by rote. Little wonder that as the children move up to the higher grades and arithmetic becomes more complex, it becomes more distasteful! It is not easy to like something that is difficult or that does not make sense.

Let's go back to the first grader. During the first three or four months of school, most of the children have learned to count at least to ten. They have had practice finding out how many children are present in their row that morning, how many children are going to buy milk at noon today, how many boys are playing with Peter's new truck. They have counted the windows in the room, the chairs and desks, the books, the jig-saw puzzles. They recognize groups of things up to ten. They have drawn pictures of three dolls, of eight cars, of five wagons. They understand one-to-one correspondence. ("Jane, will you please pass out the paper? Each child gets one piece.") They may even understand the more complicated mathematical concept as demonstrated in the direction, "We are going to share the scissors today. Two children will share one pair."

As the children talk about numbers throughout the course of the first few months, the teacher records the numbers on the chalkboard and on charts. This is spontaneous and frequently repeated. It is planned, not casually or incidentally presented. The children watch and learn. Sometimes, and here we slip into the second phase of recording number, a child is asked to record a number on the board.

Number abounds in every aspect of a child's life. The teacher should call the child's attention to this, to make him increasingly aware of number.

There are so many opportunities for this! In story the children hear about the three little pigs, the three billy goats gruff, the seven sillies. In song they count one little, two little, three little Indians. They sing about the four crows that sat upon a wall on a cold and frosty morning. They sing, "This old man, he played one. . . . " They do finger plays—"Two little blackbirds sitting on a hill, one named Jack and the other named Jill." They recite, "One, two, buckle my shoe." If you listen to their words at play you may hear things like, "Eight and eight are sixteen, wash your face with kerosene." Some first graders can count backwards from one hundred. On the school bus they join the older children in singing about bottles that accidentally fall from the wall.

So here it is about Christmastime. The children are enjoying all sorts of number experiences. Perhaps most of the children know how to write numbers from one to ten. They understand, for example, that 7 repre-

sents seven cats or seven houses etc. They form the 7 correctly. They are beginning to do their own recording.

Developing the Number Facts

Now is the time to begin the recording of facts in relation to concrete experiences. The children know that if two boys are at the library table and that if two more boys walk over to it, then there are four boys at the table. Dramatize it. Have someone record $2+2=4$ on the chalkboard. Let others take turns acting out and recording the problem. Then let them all record it on paper.

This is the third step in recording number. The children are working in the concrete. But often new addition facts are taught too soon without the benefit of concrete objects. The children are thrown into a sea of number with the chance of sinking or swimming depending upon their ability to think in the abstract. Some can and some cannot.

Why do teachers do this? There are several reasons. First of all, most of us adults do not realize the difficulty that many six-year-olds have in visualizing and thinking in the abstract. It seems perfectly simple to us that one and three are four. A child may be able to tell us four when we say, "Jack had a cookie and his mother gave him three more. How many did Jack have then?" This is a social situation. The problem $1+3=4$ is not exactly the same. It is an intellectual problem and is wholly abstract.

A second reason why teachers plunge in too fast is the pressure of time. The winter holidays have, to many adults, the awful aura of the halfway mark. "Here we are with so little time left, and there's so much work to cover," we wail with a rising feeling of panic.

Still another factor in a teacher's hurrying the children past the concrete stage is the presence of arithmetic workbooks. Many schools have workbooks available for the pupils. Too often we feel that if we have books, we must use them. Each page must be completed. In order to finish, we've got to get started and to keep plugging away at the books. We let the workbooks come between us and the children.

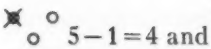
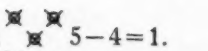
Most workbooks are written so that the work is presented in a systematic and scientific way. Too often, though, workbooks are misused. We fail to recognize that the picture of the marble is not the marble itself. the picture is semi-concrete. It is a representation, not an actuality. It is not perfectly obvious to some children that a picture of two marbles and three marbles are five marbles. So we use the workbook not as a tool for effective teaching but as an indication that a certain amount of work has been presented. The fact that workbook pages are filled, even correctly filled, does not necessarily mean that the child understands to the fullest what the work means. Used in the right way, workbooks are a valuable aid in summarizing work and strengthening concepts that have already been established. A teacher should not let the pressure of fleeting time force her into using workbooks too soon. The children may not be ready for the semi-concrete stage of number.

How to keep the work concrete? Let the children have things to count. They may have beads or beans or bottle caps or pebbles. They may use blocks or paper squares or little candies. They should manipulate objects to discover the facts they are working with. Then they can record what they have found out for themselves.

Shortly after the concept of addition is developed, the concept of subtraction should be presented. When children are allowed to work with concrete objects, they do not become confused with the various combinations. They first understand that addition is a *putting-together* process. Then they learn that subtraction is a *taking-away* process. They can, in working with concrete objects, easily see that

o and o o o are o o, $1+4=5$, and $4+1=5$.
o o o o

They should learn that this is part of a family. The rest of the family are

 5−1=4 and  5−4=1.

It is about this point that the average child is ready to move to the semi-concrete stage, the stage of pictures and charts, of flannel boards and workbook pages, of teacher-made duplicated worksheets, of little circles meaning cookies or oranges or candies or snowballs. The pictures of the marbles in the workbook make more sense now than they did before. This is the stage where little "hashmarks" appear on the chalkboard or the paper beside each algorism. It is the transitional stage toward work in the abstract, and it may be a very brief period for some children.

By this time (perhaps about the beginning of the second semester) the teacher will find that some children are quite ready to solve problems completely in the abstract. This is the fourth and final step in recording number. The other steps, if we may be allowed to recapitulate, are those of the teacher recording, of the children observing and learning and practicing, and of recording number facts in relation to concrete experiences. At the same time that some children find themselves ready to work in the abstract, others may have to return to the semi-concrete to discover the answers to new number combinations. Some children may have to manipulate their counters, working at the concrete stage, in order to record the answer.

Grouping is Helpful

About this time the teacher is going to have to do something to take care of the children who are eager to go ahead. Something else must be done for those who cannot go so fast. Here the teacher may find that grouping the children is the answer. Most teachers group the children for reading. Dividing the children into groups makes for more effective teaching in arithmetic, too, some teachers find. How many groups a teacher has depends upon the class, its size, makeup, and progress. Most teachers work with three reading groups. It is possible that in arithmetic the teacher may need three groups, one for working on the concrete level, one for those working with the semi-concrete, and one group for the children working in the abstract.

Perhaps the class divides itself into only two groups. The groups, of course, need not be the same size. Possibly a larger group might be composed of those children working in the abstract and the semi-concrete levels while the smaller group is made up of the children still working in the concrete stage. This last method of grouping is likely to stimulate some of the children working at the semi-concrete level to abandon their "crutches" a little more quickly because they see some of their friends working without them. No pressure is put upon them otherwise, but the grouping is structured so as to encourage growth to recording in the abstract. The teacher should not worry about undue competition among the children if the emphasis is placed on accuracy and if there is a genuine atmosphere of understanding and tolerance for differences in individual rates of progress.

We often think of teaching as an art. We speak of the creative teacher, and that phrase evokes the artistry involved in teaching. However, there are also scientific aspects in teaching. The psychology of learning is a science. Three experiments are suggested here for teachers interested in applying the scientific method to their teaching:

1. Provide plenty of concrete materials for children to manipulate.
2. Give the children plenty of opportunity to use these materials.
3. Group the children for arithmetic.

These ideas are not really new. They have been tried, and they have been found to be of value. For the teacher who has not yet tried these suggestions, this will be experimenting. It should be a stimulating experience to watch the results. Some of the ques-

[*Concluded on page 234*]

Developing Creativity through Arithmetic

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WHEN MAKING ARRANGEMENTS for taking her fourth grade class through a local manufacturing plant, Miss Bliss was told that plant guides would take a group of eight children at a time through the plant. Since Miss Bliss had become concerned about whether she was developing creativity, she asked each member of her class to figure out how many groups of eight would be formed from the class of thirty-two children, and to work out this problem in as many different ways as they could. In looking over the papers handed in by the children, Miss Bliss found seven different ways in which the answer was correctly found:

1. $32-8=24,\ -8=16,\ -8=8,\ -8=0$
   $1\qquad 2\qquad 3\qquad 4$; 4 groups
2. $32\div 8=4$ groups
3. $8\overline{)32}$
   4 groups
4. $8+8=16,\ +8=24,\ +8=32$
   $1\ \ 2\qquad 3\qquad 4$; 4 groups
5. 5 groups of $8=40-8=32$
   5 is one group too many
   4 groups
   $2+2=4$ groups
6. $8\overline{)32}$
   16
   —
   16
   16
   —
   0
7.
   1 group
   2 groups
   3 groups
   4 groups

Miss Bliss realized that she could not be sure of how individual children might have proceeded if they had not been asked to work the problem in different ways, but in looking just at the work handed in she could see four levels of arithmetic understanding and application:

1. Absolute recall, as apparently seen in the second example. The child had the answer at his fingertips almost as soon as he comprehended the problem.
2. Application of rules as seen in the third example. The child saw the problem and drew the rule from his memory and then proceeded to solve the problem. Miss Bliss knew, however, that the use of rules is not always successful. Too often children said, "Yes, I can work the problem. Do you multiply or divide?" Therefore, Miss Bliss designated subcategories for this level of operation:
   (a) Application of rules with understanding of the conditions under which the rule can be used
   (b) Application of rules by rote memory
3. Understanding of meaning, but used low level skills or lacks certain skills, as in examples 1, 4, 5, and six. The child understood the problem to be solved, but did not use the most efficient computational method.
4. One-by-one counting used, but understanding present as in example seven. Although this was a low level of operation, Miss Bliss could think of no problem in arithmetic that can not be solved by counting as long as the understanding of what to count is present and time is available.

Miss Bliss had become interested in the problem of making arithmetic a creative experience when she read an article concerning a course offered at Massachusetts Institute of Technology for prospective engineers in which they were asked to invent tools for mythical people in a mythical world in which the physical features of the people and the world were entirely different from our world. The article described the difficulty that these college students had in discarding

their preconceived notions of the "best" and the "right" way of doing things and of being creative in their thinking about engineering.

A trip to the dictionary showed Miss Bliss that to create is, " . . . to make, as a new product, out of existing materials, newly or differently assembled . . . ." Did this definition apply only to activities such as art, music, or dramatics? Could it be applied to arithmetic? Miss Bliss looked for the dictionary definition of arithmetic and found, "The *science* of numbers; the *art* of computation or reckoning by figures." After some consideration Miss Bliss concluded that arithmetic could be considered a creative activity if it could be properly taught, and that she was in even a better position than were the professors at M.I.T. since her children had not had time to build up so many stereotyped ideas about right ways and wrong ways.

"But were my children really creative today, or were they just being different?" Miss Bliss asked herself. "Surely just doing things differently does not mean being creative." As Miss Bliss thought the matter through she decided that while being different did not necessarily make one creative, it is extremely difficult to be creative without being different. Undoubtedly some of the thirty-two children who worked on the arithmetic assignment this morning were not having a creative experience but were merely trying to be different to please Miss Bliss. Others, however, had really seemed to be testing out their own ideas and procedures—experimenting, accepting and rejecting solutions. Certainly individual differences would have to be taken into account if arithmetic were to provide a creative experience for all the children.

"And some of those solutions certainly took a long time," thought Miss Bliss. "Billy got the right answer by counting by ones, and his solution may show understanding and creativity, but surely he needs more specific skills than he has now." After thinking over this problem Miss Bliss realized that some of her time spent in teaching arithmetic creatively would have to be devoted to letting children compare and evaluate the different procedures used. It seemed probable that individuals would quickly become aware of questions of efficiency and would implement meaningful concepts with economical methods of computation. In many cases, Miss Bliss was sure, children would evolve to the same basic procedures which Miss Bliss now taught, but these procedures would be evolved after exploration and after meaning had been established. Even more encouraging would be the students who found procedures different from those previously taught—procedures superior to those in Miss Bliss's arithmetic text books.

Reasons for Creativity

As she summarized her thinking Miss Bliss could see three basic reasons why she should attempt to develop creativity in arithmetic. (1) Learning to be creative in arithmetic will likely serve to further the development of creativity in general. Miss Bliss could not conceive of learning to draw without drawing or learning to act without acting. In the same way, she could not conceive of an individual being creative without many specific opportunities for creativity in as many different areas as possible.

(2) Learning to be creative in arithmetic will serve to promote more complete understanding of arithmetic concepts. "I don't like arithmetic. It's too hard!" was a frequent comment of some of Miss Bliss's pupils. But was it the difficulty of the subject or the lack of meaning in the mind of the child that caused the distain of the pupil? Miss Bliss remembered the confused and resentful children in her class last year when she attempted to show them how to divide 623 by 7. Her explanation had been carefully planned and exact, with the problem broken down into four steps: (1) "Seven will not go into 6;" (2) "Seven will go into 62 eight times"; (3) "Put the eight over the two and multiply eight times seven, write the answer under 62 and subtract"; (4) "Bring down the 3 and put seven into 63 nine times. Place the nine over the three."

Her ears still burned when she remem-

bered how Mary had stopped watching her explanation after the first step and started writing on her paper. As soon as she had finished Mary's hand shot up and she said, "We learned last year that the six in the hundreds' place stands for 600 and 7 will go into 600 85 5/7 times, so is it right to say that seven won't go into 600?" Miss Bliss was sure that no one could have understood her stuttered explanation. Perhaps she had developed insight along with the children. They now understand that division means separating a quantity into equal parts.

(3) Learning arithmetic creatively provides a sure way of demonstrating that understanding of arithmetic concepts has occurred. Miss Bliss realized that the children who had experimented with today's assignment—the children for whom it had been creative, showed in their work an easily grasped picture of their thought processes and their level of understanding.

Tomorrow three of Miss Bliss's pupils are scheduled to demonstrate the solution of the same problem that Miss Bliss struggled with last year. At noon today each of them demonstrated to Miss Bliss how he had worked out his solution. One had used an abacus and had worked out procedures to transfer equivalent values from the hundreds column to the tens column, to the ones column. Another child had worked out his solution by using a place value chart, while another child drew on his knowledge of multiplication to solve the problem. As he explained to Miss Bliss, "I know there are seven groups of seven in 49. That means 14 groups of 7 in 98, and 98 is almost 100. There are six groups of 100 in 623, so I multiply 14 times 6 and get 84. I have two ones left over in each of the groups of 100, and I have 23 more ones besides. Twenty three plus 12 gives me 35 to divide into groups of seven, and I know that $7 \times 5 = 35$. Then I add 84 and 5, and I know that there are 89 groups of seven in 623."

There was no question in Miss Bliss's mind that these three children understood the meaning of division, and with their present insight she was not concerned about the pupils' ability to see quickly the advantages of more efficient computational procedures. Certainly it was important that children find the best ways of solving problems that have been evolved to the present time. But Miss Bliss was convinced that our present best knowledge might not fit the changing world of the future. To her the value judgment seemed clear—the best preparation that she could give children for an uncertain world would be to develop their creative abilities to the utmost. Not answers, but a way of finding answers; not what to think, but how to think—children with these qualities are the hope of the future.

EDITOR'S NOTE. Should we have "Creativity Through Arithmetic" or "Arithmetic Through Creativity"? Messrs. DeLong and Clark use *creativity* in much the same sense as other authors have been advocating *discovery*. The essential element is that children shall participate in the steps of learning with independent thought instead of learning to re-cite the learning of others from memory. This creativity or discovery yields an independence to a child in that he can fall back upon an understanding and recreate instead of relying upon memory. It is extendable and applicable to other areas for discovery. When pupils explore, create, and discover several different methods of work will be represented in a class. Then the teacher must decide if he will work toward having all pupils using the same method of work and when it is opportune to do this.

Grossnickle

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pupils. In each of the three stages in the development of the multiplication facts, emphasis has been placed on learning which results from discovery.

EDITOR'S NOTE. Professor Grossnickle shows how to use several levels of discovery but he is not content with just discovery of facts, he emphasizes also the discovery of mathematical relationships. It is this working with relationships that fosters insight that may have far-reaching effects with certain youngsters. It is heartening to note how arithmetic is changing from the routine memorization of facts and processes to an educational experience wherein each may explore under competent guidance and be a genuine learner. Machines may serve us as robots for mechanical tasks but we are always in need of people who understand and who can think and hence direct the mechanical operations.

Per Cent Without Cases

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ONE OF THE MORE DIFFICULT topics for arithmetic students is per cent. We feel that one of the main reasons for this difficulty is the arbitrary division of per cent into the three cases. Even in many of the newer texts per cent is treated in the same old way; however, there are now fortunately some exceptions. But since by far the great majority of the texts now in use treat per cent in a very mechanical way, it may be of help to show that there is only one rule involved in the study of per cent, not three.

We have used the following approach in classes of remedial arithmetic and algebra for college freshmen and have found its results more satisfying than the "case" approach. We do not claim the method is a panacea which will forever remove all difficulty with per cent for everyone.

It is not intended that what is presented here should be lifted bodily to the sixth and seventh grade classroom without first being adapted to fit the backgrounds and abilities of the students being taught. We do hope the following discussion will help clarify the concept of per cent for teachers, so that they may create their own adaptations and teaching devices for classroom use.

As soon as the student knows how to multiply and divide common fractions, he is ready to do per cent problems. By careful choice of the numbers involved one can do a sufficient number of problems in per cent, using the arithmetic of common fractions only, to make per cent a worthwhile application of fractions. Admittedly, knowledge of decimal fractions simplifies the arithmetic, but decimal fractions are not absolutely necessary for the study of per cent at the beginning.

Our first order of business is to develop a clear notion of what we mean by "per cent." We feel a mathematical definition has definite advantages over the usual words mentioning "parts per hundred." By *A is P per cent of B* we shall mean these three numbers satisfy the following relationship:

$$\frac{P}{100}=\frac{A}{B}.$$

This is a formula just as are $A=1\times w$, $A\times\pi r^2$, $d=rt$, etc. If any two of the three letters are assigned values, then the third letter can be found by substituting in the formula and performing arithmetical operations on the formula. We shall illustrate by working an example of each of the three cases. Until the students are familiar with decimal fractions, it would probably be advisable to consider only problems in which all the numbers involved are integers.

Example 1: What number is 30 per cent of 60? Or, as usually stated, find 30 per cent of 60.

Solution: We shall frequently want to make use of the words "is equivalent to" or "means the same thing as," so we shall introduce the symbol $\leftrightarrow$ to represent either of these phrases. Our first step is to write the following:

$$A \text{ is } P \text{ per cent of } B \leftrightarrow \frac{P}{100}=\frac{A}{B}.$$

Our problem now is to determine which of the numbers in the formula we know and which we do not know. The key to this problem is the phrase "30 per cent of 60." At the risk of being repetitious we write

$$A \text{ is } P \text{ per cent of } B \leftrightarrow \frac{P}{100}=\frac{A}{B}$$

$$A \text{ is } 30 \text{ per cent of } 60 \leftrightarrow \frac{30}{100}=\frac{A}{60},$$

and we see that it is A we do not know. Our problem now is to determine what value A must have in order to satisfy the relationship. We notice that if we multiply $A/60$ by 60 we get A. This suggests we perform the following arithmetical operation:

$$\frac{\cancel{60}^{\,1}}{1}\times\frac{\cancel{30}^{\,\cancel{10}^{\,1}}}{\cancel{100}^{\,\cancel{60}^{\,1}}}=\frac{A}{\cancel{60}^{\,1}}\times\frac{\cancel{60}^{\,1}}{1},$$

or

$$18=A.$$

If the students know how to determine whether two given fractions are equal, this answer should be checked by seeing whether

$$\frac{30}{100}=\frac{18}{60}.$$

If some observant student should ask why both sides of the equality were multiplied by 60, what should be your answer? An example or two showing that it only one of two equal fractions is multiplied by a number, then the equality is destroyed should convince the student of the necessity of multiplying both sides of the equality. Further examples would also convince him the same holds true for division, addition, and subtraction. For example,

$$\frac{3}{4}=\frac{6}{8},$$

but

$$\frac{2}{1}\times\frac{3}{4}\neq\frac{6}{8}.$$

However,

$$\frac{2}{1}\times\frac{3}{4}=\frac{2}{1}\times\frac{6}{8}.$$

Finally the class should agree that one could formulate the following rule: If two numbers are equal to each other, then the new numbers obtained by multiplying or dividing each of these equal numbers by the same number will be equal, also. Further, if two numbers are equal to each other, then the same number may be added to or subtracted from each of these numbers without destroying the equality. We should not make the mistake, however, of saying we "proved" these rules. The impossibility of division by zero may be mentioned if the group is capable of appreciating this.

After this digression let us proceed to the next two examples.

Example 2: What per cent is 17 of 34? Or, 17 is what per cent of 34? We proceed as before by writing

$$A \text{ is } P \text{ per cent of } B \leftrightarrow \frac{P}{100}=\frac{A}{B}$$

$$17 \text{ is } P \text{ per cent of } 34 \leftrightarrow \frac{P}{100}=\frac{17}{34}$$

By analogy to the preceding example we see that multiplying both sides of the equation by 100 will "solve" the equation for P, or will "isolate" P.

$$\frac{P}{\cancel{100}^{\,1}}\times\frac{\cancel{100}^{\,1}}{1}=\frac{17}{34}\times\frac{100}{1}=\frac{1700}{34}=50.$$

Or $P=50$. This again should be checked.

Example 3: 15 is 20 per cent of what number. Or, find a number 20 per cent of which is 15.

As before we write

$$A \text{ is } P \text{ per cent of } B \leftrightarrow \frac{P}{100}=\frac{A}{B}$$

$$15 \text{ is } 20 \text{ per cent of } B \leftrightarrow \frac{20}{100}=\frac{15}{B}$$

In this situation B cannot be determined in one easy step. Our first task is to remove B from its role as a denominator of a fraction. By looking at the two preceding ex-

amples we notice that in each instance multiplication of the equation by a denominator resulted in removing this number from its role as a denominator. In Example 1 this number was 60, and in Example 2 it was 100. By analogy we should try multiplying both sides by B.

$$\frac{20}{100}\times\frac{B}{1}=\frac{15}{\cancel{B}}\times\frac{\cancel{B}}{1}=15.$$

or

$$\frac{20\times B}{100}=15.$$

Now the equation looks somewhat more like those in the preceding examples. At least we know how to disassociate the 100 from the B; we multiply by 100, giving

$$\frac{20\times B\times\cancel{100}}{\cancel{100}}=15\times 100.$$

Or, upon simplifying,

$$20\times B=1500.$$

This equation says in effect that 20 times some number is 1500, and we want to know the number. Suppose we give the same problem a more simple form such as

$$3\times B=12.$$

This says 3 times some number is equal to 12, and we want to know the number. The students will give the correct answer in this case. Now let us see if there is some arithmetical operation we can perform on $3\times B=12$ which will result in $B=4$. After some reflection someone will surely come up with, "Divide by 3." By analogy the students should then be lead to see that $20\times B=1500$ should be solved for B by dividing by 20, giving

$$\frac{\cancel{20}\times B}{\cancel{20}}=\frac{1500}{20}.$$

Or

$$B=75.$$

This answer should, of course, be checked in the original equation or formula.

Remark: We are quite aware that what we have been indulging in in the preceding is informal algebra, but we prefer to call it generalized arithmetic. If your administrators or the parents of your students are allergic to "algebra," then do not use this objectionable word. There is no doubt that much informal algebra and geometry will be in the upper grade arithmetic curricula in time to come, and we feel that the type of arithmetic used above is not above sixth graders who know, for example, how to multiply $\frac{3}{8}$ by 4 or who can see relatively quickly that $\frac{3}{4}\times 4=3$.

While we are digressing, it may not be amiss to give some hint of some of the basic algebraic principles which have been used above and with which the teacher of a sixth grader—but not necessarily the sixth grader—should be familiar.

1. Simple equations such as we deal with in per cent problems are solved by arithmetical operations, not by black magic or by guess.
2. The kind of operation needed is indicated by the equation:
   A. If $3\times N=12$, then we can solve for N by *dividing both* sides by 3, giving $N=4$.
   B. If $N/6=12$, then we can solve for N by multiplying both sides by 6, giving $N=72$.
   C. If $N+4=12$, then we subtract 4 from both sides to give $N=8$.
   D. If $N-5=12$, then we add 5 to both sides giving $N=17$.

We asserted the type of operation needed is indicated in the equation, and so it is.

There is one property the operations used in the four types above have in common. We note the following:

Equation	Indicated Operation on N in equation	Operation used on equation to find N
$3\times N=12$	multiplication by 3	division by 3
$N/6=12$	division by 6	multiplication by 6
$N+4=12$	addition of 4	subtraction of 4
$N-5=12$	subtraction of 5	addition of 5

We see that only numbers in the equation and operating on N are used on the equation. Secondly, the operation used is the opposite of that indicated in the given equation.

It is true that an equation such as

$$\frac{12}{N}=3$$

must first be put into the first of the forms above by multiplying by N. But since in the equation N is acting as a divisor, even here the operation of multiplying by N is opposite.

Fundamental in any operation on equations is the absolute necessity of performing a given operation on both sides of the equality sign.

After this lengthy and probably unnecessary digression we return to the consideration of per cent. After students have knowledge of decimal fractions, we can change the form of the rule for defining per cent to make better use of this tool. We recall

$$A \text{ is } P \text{ per cent of } B \leftrightarrow \frac{P}{100}=\frac{A}{B}.$$

We notice that $P/100$ can be changed to a decimal fraction simply by dividing P by 100, i.e., by moving the decimal point in P two places to the left. For example,

$$25 \text{ per cent}=\frac{25}{100}=.25$$

$$135 \text{ per cent}=\frac{135}{100}=1.35$$

and

$$.75 \text{ per cent}=\frac{.75}{100}=.0075.$$

In the above the numbers in the left hand column were the P's in the formula

$$\frac{P}{100}=\frac{A}{B},$$

while the right hand column is the decimal form of P. In order to facilitate matters let us agree that

$$\frac{P}{100}=p.$$

Then in referring to the above numbers we would have

P per cent	*p (P in decimal form)*
25 per cent	.25
135 per cent	1.35
.75 per cent	.0075

Our preceding definition and formula can then be written as follows:

$$A \text{ is } P \text{ per cent of } B \leftrightarrow p=\frac{A}{B}$$

where p is decimal form of P. Or we could write, after multiplying both sides by B,

$$P \text{ per cent of } B \text{ is } A \leftrightarrow p\times B=A$$

where p is the decimal form of P. We have changed the word order above but the meaning remains unchanged. We now use this last form to illustrate problems such as were illustrated previously.

Example 4: Find 27 per cent of 42. Or, what number is 27 per cent of 42?

Solution:

P per cent of B is A

$\leftrightarrow p\times B=A$ (p decimal form of P)

27 per cent of 42 is A

$\leftrightarrow .27\times 42=A,$

or

$$11.34=A$$

Example 5: 20 per cent of what number is 8.1?

Solution:

P per cent of B is A

$\leftrightarrow p \times B = A$ (p decimal form of P)

20 per cent of B is $8.1 \leftrightarrow .20 \times B = 8.1$.

The equation $.20 \times B = 8.1$ is exactly the same type as $3 \times B = 12$ and is solved the same way. Hence, we must divide both sides by .20 giving

$$\frac{.20B}{.20} = \frac{8.1}{.20},$$

or

$$B = \frac{8.1}{.20} = 40.5.$$

This answer should be checked by taking 20 per cent of 40.5.

Example 6: 64 is what per cent of 32?

Solution:

P per cent of B is A

$\leftrightarrow p \times B = A$ (p decimal form of P)

P per cent of 32 is $64 \leftrightarrow p \times 32 = 64$.

We divide both sides by 32 giving

$$\frac{p \times 32}{32} = \frac{64}{32},$$

or

$$p = 2.$$

Since $p = P/100$, $P = 100 \times p$, so that in this problem P $100 \times 2 = 200$ per cent.

Remark: All too many students would get an answer of 50 per cent in this problem.

Some textbooks also leave the impression that there are four different cases of simple interest problems represented by the four formulas:

$$I = prt, \quad r = \frac{I}{pt}, \quad P = \frac{I}{rt}, \quad t = \frac{I}{pr}.$$

Since all of these are simply different forms of $I = prt$, there is no reason to remember any other formula than this. For example, consider the following typical problem.

Problem: For how long did I borrow \$1000 at 6 per cent simple interest if the interest I paid amounted to \$90? We substitute the values we know in the formula:

$$I = prt$$

$$\$90 = \$1000(.06)t$$

This can be simplified by multiplying the \$1000 by .06 to obtain

$$\$90 = 60t.$$

This equation is of the same type as

$$3N = 12$$

which is solved by dividing both sides by 3, the multiplier of N. Hence, in the present case we divide both sides by 60, the multiplier of t, resulting in

$$\frac{\$90}{60} = \frac{60t}{60},$$

or,

$$\frac{3}{2} = t.$$

That is, we had the money for one and one-half years.

Before closing perhaps we should show how the generalized arithmetic illustrated previously can be applied to advantage in a somewhat more complicated problem.

Problem: A business pays \$10 for an item and wants to sell it at a price resulting in a profit of 30 per cent of the selling price. What should be the selling price under these circumstances?

The basic principle involved here is that cost plus profit equals selling price, at least

in textbook problems. We can represent this rule by the formula

$$C+P=\text{S.P.}$$

If we try to substitute in this formula, we find we are given only the cost explicitly. We, therefore, try to express P in terms of S.P. and thereby eliminate P. The problem states the profit is 30 per cent of the selling price. The phrase, "30 per cent of," should remind us of

$$A \text{ is } P \text{ per cent of } B \leftrightarrow p \times B = A$$

where p is the decimal form of P. Note that we are using P to represent both profit and per cent. Then, by analogy to the above, profit is

$$30 \text{ per cent of S.P. is } P \leftrightarrow .30 \times \text{S.P.} = P$$

So for P in $C+P=\text{S.P.}$ we substitute $.30\times\text{S.P.}$ and \$10 for C giving us the new equation

$$\$10+(.30\times\text{S.P.})=\text{S.P.}$$

Now in this equation there is only one unknown quantity, S.P. We proceed to get all the terms involving S.P. on the right side of the equation. To do this we must subtract $.30\times\text{S.P.}$ from both sides since this is the opposite of the indicated operation in the equation. This will give us

$$\$10=\text{S.P.}-.30\times\text{S.P.}$$

The right side of this equation now instructs us to subtract three-tenths of the selling price from the selling price. This would result in seven-tenths of the selling price on the right side. Or

$$\$10=.7\times\text{S.P.}$$

This equation is of exactly the same type as

$$12=3\times N$$

and is solved in the same manner. Accordingly, we divide both sides by .7 and obtain

$$\frac{\$10}{.7}=\frac{.7\times\text{S.P.}}{.7}$$

or,

$$\$1.43=\text{S.P.}$$

to the nearest cent.

We hope some of the readers will have the courage to teach per cent without cases, departing from the text, if necessary, to do so. We do not recommend using the word "algebra" in sixth or seventh grade at present, but we do recommend using the informal algebra illustrated in the examples. We feel certain it will be less confusing and certainly more systematic than trying to distinguish between the three cases. We feel substituting in the formula used as the definition of per cent will result in better understanding of the concept of per cent, and that the arithmetical operations used on this formula are quite natural.

EDITOR'S NOTE. Mr. Wendt presents a logical structure and argument. We may not like it but before dismissing a new procedure let us remember that we are all creatures of our early training and habits and we may find it difficult to be objective about a new approach. If one is to use procedures that we recognize as essentially algebraic, why not go a step further and learn to solve the basic proportion and then use this procedure on Mr. Wendt's central equation

$$\frac{P}{100}=\frac{A}{B}?$$

Or why not use the more common; Base×Rate = Percentage or $P=BR$?

In viewing percentage as a topic in school, business, and trade we find the one important concept to be "100% of anything represents the whole of that thing" and from this we may proceed variously. Pupils can build several different procedures to solve the "cases." Probably the more common and easily understood case is number 2, expressing a relationship as a per cent. e.g. 3 is what per cent of 12?

What do readers think of Mr. Wendt's approach? Is it good for a teacher's understanding? Is it good for pupils in grades six and seven? Does it result in better understanding and more facile work? Is it easier to use with the many problems that are found in the general category of percentage? Should we teach only the simple items in percentage to be solved by arithmetical procedures and then concentrate on the topic after the basic axioms of algebra have been established? How will the basic formula be established? If one does this inductively toward the generalization, why not continue an inductive approach throughout the topic?

A Trip to Notation Land

MARY GERALDINE GREEN
Mercy College, Detroit, Mich.

I. All Aboard for Egypt

DOES YOUR ARITHMETIC CLASS need a new zip and zing? Well, fifty-eight fourth graders and fifty-two second graders found this zip and zing that can make arithmetic become more meaningful and alive through the study of this science of several centuries ago—when the people of Egypt and Greece realized that they needed a system of counting over and above the primitive man's method of one-to-one correspondence. Thus we find that these one hundred ten children in Detroit, Michigan have arrived recently at modern concepts of arithmetic through their brief travels back to the days of ancient Egypt and Greece. These pupils were conducted on their tour by Sister M. Alphonse and Sister M. Charitas who were pursuing a course of College Arithmetic. In this course, principles of arithmetic are taught for the purpose of giving future elementary teachers a firm background in arithmetic, and we also concern ourselves with the actual teaching of arithmetic. As a result of this, Sister Charitas is catching the express to tell how she helped her second graders understand zero being used as a place holder. She accomplished this through the comparison of the Hindu-Arabic system of notation with Egyptian methods of notation. Sister is very eager to relate how she cautiously proceeded to introduce them to the following symbols:

	Egyptian Notation	Hindu-Arabic
|	vertical staff (Used for one to ten expressed by repetition)	one
∩	heel-bone (Used for ten to one hundred)	ten
9	scroll (Used for one hundred to one thousand)	one hundred

We offer the example of 243 which would be shown as:

99 ∩∩∩∩ |||

Blackboard drill was used to strengthen the concepts of one and ten, but trouble sprang up for the second graders, as an obstacle upon the track, when they tried to write "teen numbers" in our number notation. Some admitted that it was because of the way our "teen numbers" are pronounced: an example giving proof of this is the word "sixteen," for one child stated that he heard the six first so he would write 16 as 61. However, after learning that Egyptian symbols follow a pattern from left to right of hundreds, tens and units, then he could easily put sixteen in its proper order for the Egyptian system—thus giving them a lesson in place value. Therefore, the next step would be to show that zero shows place value also. Since the youngsters knew how to form 203 in the Egyptian form:

99 |||,

through careful teaching the little minds could grasp the meaning of zero as a place holder in such a number by showing that a missing symbol in Egyptian form is met by a 0 in Hindu-Arabic notation. Sister claims that this solved all of her problems in trying to get across the ideas concerning zero as a place holder—thus she is mighty proud of these results. Three classes of drill were held, after which pupils admitted that they really had fun, but at the same time the teacher states that it was a real challenge for them and thus I offer her own words to substantiate the use of this system today:

One of the numerous ways to counteract boredom and stimulate interest in arithmetic would be to resort to an earlier method of number writing.

Young minds must be constantly challenged and motivated. Consequently, if a teacher employs such a fruitful solution which is simultaneously novel and instructive, the pupils will more readily respond and maintain their attentional span throughout the entire arithmetic period.

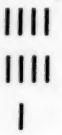
II. Last Stop, Greece

Then we get aboard Sister Alphonse's train on Track #4 and head for Greece. Here we see that Sister began also like Sister Charitas and discussed the methods of primitive man for counting and then proceeded on to the written records—first to Egypt, where the notation studied by the second graders was pursued along with two additional symbols:

Lotus flower for 1,000

Pointing finger for 10,000

After much drill, papers showed almost one hundred per cent correct results and the errors that did exist came about because some students forgot that no more than four symbols can be put into each row. Ex. 9 is



in Egyptian notation. Next, Sister presented her class with a bigger challenge by escorting them through the awkward system of Greek notation where 24 Greek letters, 3 obsolete letters and M stand for various numbers. After practicing the formation of our numbers in the Greek system, students began to realize place value existing in both systems. Ordinarily in classes such as these, children would not be expected to memorize these Greek symbols as they would those for the Egyptian system, but these children wanted to do so, and thus they learned a few at a time. After all was said and done the fourth graders preferred Greek notation over the Egyptian since there is less to write and one used less paper. On the fourth day of his new adventure, Sister presented a drill where ten numbers were to be put into Greek notation and ten in Greek notation were to be put into Hindu-Arabic form. From the fifty-eight pupils there were five perfect papers, twenty-four with only one wrong, and the rest had slight errors. Thus we conclude this portion of the trip with a statement from Sister Alphonse:

This experience created a more intense desire for learning, and alleviated the teaching of arithmetic which at times becomes a mechanical burden to some children. Our journey was indeed a pleasant one.

Thus I offer the above results achieved through the efforts of my students with the hope that this may give a new lift to some arithmetic classes, and inspire some daring teachers to attempt a correlation in arithmetic work such as we have found in our trip to Notation Land.

Eliminate "Borrowing" in Subtraction

MAX N. MOSSIL
Fulton, Missouri

A NEW METHOD which I shall call the *raised subtrahend* method obviates the need of borrowing, a leading cause of subtraction failures. It was tried for the first time this past year with amazing results. The method invokes a principle that equal raises for two numbers do not affect their differences. Accordingly, in an example such as

$$\begin{array}{r} 9\ 17/24, \\ 3\ 23/24 \\ \hline \end{array}$$

if the mixed number in the subtrahend is to be changed to a higher whole number, the minuend must be increased by as much as the raise given to the subtrahend to maintain their constant difference. The net result is that only whole numbers are directly involved in the subtraction. The algorisms for the current method and the "raised subtrahend" are illustrated.

CURRENT METHOD

A. 9 17/24 = 8 41/24
3 23/24 = 3 23/24
5 18/24
or 5 3/4

RAISED SUBTRAHEND METHOD

9 17/24 = 9 18/24 or 3/4
3 23/24 = 4
5 3/4

"23" cannot be subtracted from "17." Therefore, 3 23/24 must be raised 1/24 because it lacks this much of being 4. The 1/24 is also added to 9 17/24 to make 9 18/24. The subtrahend always becomes a whole number, thus making the subtraction of fractions unnecessary. The reduction of 18/24 is done either as shown or mentally, and the answer is cleaner than otherwise.

B. 5 = 4 12/12
3 7/12 = 3 7/12
1 5/12

5 = 5 5/12
3 7/12 = 4
1 5/12

As 5/12 is needed to change 3 7/12 to 4, this much must also be added to 5 for equal raises. Again, the subtrahend, unencumbered with a fraction is handily subtracted from 5.

Actually superior students get the answers quickly. They think: 9−4 and 1+16 instead of 8−3; 24+17 and 41−23.

The raised subtrahend method works just as effectively in the subtraction of compound numbers. The smaller unit in the subtrahend is raised sufficiently to be absorbed by the larger unit, and such a value is added to the smaller unit in the minuend. Again, the comparison of the methods:

CURRENT METHOD

A. 8 hr. 45 min. = 7 hr. 105 min.
3 hr. 55 min. = 3 hr. 55 min.
4 hr. 50 min.

RAISED SUBTRAHEND METHOD

8 hr. 45 min. = 8 hr. 50 min.
3 hr. 55 min. = 4 hr.
4 hr. 50 min.

In changing 3 hr. 55 min. to 4 hours flat, the 5 minutes are added to 45 minutes.

B. 9 ft. = 8 ft. 12 in.
2 ft. 10 in. = 2 ft. 10 in.
6 ft. 2 in.

9 ft. 9 ft. 2 in.
2 ft. 10 in. = 3 ft.
6 ft. 2 in.

As 2 more inches are needed to change '10" to 3', this amount is added to 9', as shown.

There is no doubt about the ease of mental computation. The answer can be produced in almost a single sweep: 4 from 8, and the lack of 5 minutes plus 45. It also is a short-cut in figuring the elapsed time between 7:35 A.M. and 3:55 P.M. The usual method of solution is: 7 hours 35 minutes taken from 12 hours (noon) = 4 hours 25 minutes plus 3 hours 55 minutes (after

noon) = 7 hours 80 minutes, or 8 hours 20 minutes. The faster way would be to raise 7:35 to 8:00 and advance 25 minutes in the other time to make it 4:20. Then, it's 12 − 8 = 4; + 4:20 = 8 hours 20 minutes.

The raised subtrahend method is just as workable in the subtraction of whole numbers, and it is ideally used for students weak in higher number facts and in borrowing. One will find it a very rewarding teaching tool, for all number facts whose minuends exceed 10 are excluded, leaving smaller number facts to be absorbed in memory work. The key is knowing the combinations: 10 − 9, 10 − 8 · · · down to 10 − 1 that makes the raised subtraction possible.

Initial teaching is perforce confined to the minuends ranging between 11 and 18 such as

$$\begin{array}{r} 1\,3. \\ 6 \\ \hline \end{array}$$

The rationalization, although not absolutely necessary, would run more or less like this: Since we cannot take 6 from 3, we raise 6 to 10, the full unit, and also raise 3 in the minuend by as much as 6 was raised to make 10

13 would in a sense become	17.
6	10
—	—
7	7

However, we need not think the change-over in this way. Preferably, the inner speech is: 4, the complement of 6 in 10, is *added* to 3 to make 7. In

$$\begin{array}{r} 14, \\ 8 \\ \hline \end{array}$$

2 (the complement of 8) is added to 4 to make 6. In other words, complementary numbers for such subtraction are two digits whose sum is 10. This method may result in a slight reduction in speed but it certainly provides for a much greater accuracy. In any event, speed in the adult life will deteriorate more or less while accuracy will become an enduring quality. It must be added that this program of subtracting whole numbers should not be forced on children already skilful in the conventional way. Rather, it—if it is to be adopted at all—should begin with the students learning subtraction for the first time.

For a more difficult example 642 − 187, the raised subtrahend method can be employed in either the equal-additions method or the decomposition method to a very good advantage.

RAISED SUBTRAHEND—EQUAL-ADDITIONS METHOD

6 4 2 1 8 7 —— 5	7 cannot be taken from 2, so 3—the complement of 7 in 10—is added to 2 to make 5.
6 4 2 9 1 8 7 —— 5 5	Use "1" in "10" to raise 8 to 9. 9 cannot be taken from 4, so 1—the complement of 9 in 10—plus 4 = 5.
6 4 2 2 9 1 8 7 —— 4 5 5	Use "1" in "10" to raise 1 to 2. 2, less than 6, is unchanged and subtracted to make 4.

RAISED SUBTRAHEND—DECOMPOSITION METHOD

6 4 2 1 8 7 —— 5	We cannot take 7 from 2, so instead 3—the complement of 7 in 10—is added to 2 to make 5.
3 6 4 2 1 8 7 —— 5 5	Use "1" in "10" to lower 4 to 3 because 42 − 10 = 32. 2, the complement of 8 in 10, plus 3 = 5.
5 3 6 4 2 1 8 7 —— 4 5 5	Since a complement was used, lower 6 to 5. No complement for 1, and it is 5 − 1 = 4.

Let's see what happens when a child unnecessarily or dreamily uses the complement of a subtrahend already smaller than the minuend:

8 7 9 3 5 6 ——	4, the complement of 6 in 10, is added to 9 to make . . . oops! . . . 13. Dreaming again, stupid me!

Practically fool-proof, we may say. Yet, the difficulties often arise when the minuend contains a series of zeros for the decomposition method, and to a lesser extent when the subtrahend has a series of nines for the equal-additions method. It can be safely believed that the raised subtrahend method will resolve the difficulties for the newer generation if they don't have to learn anything about the meaning of borrowing. If the decomposition method won't be given up without a fight, then the suggested solution is this:

```
8 0 0 0   8 cannot be taken from 0, so 2—the comple-
2 6 5 8   ment of 8 in 10—plus 0=2.
---------
      2

7 9 9
8-0-0 0   Use "1" in "10" to count down 800 to 799
2 6 5 8   because you cannot lower a "0" alone.
---------
```

In any event, the auxiliary figure "1" is never used thus resulting in a very clean and effortless work—unlike for the decomposition method alone. All told, fewer number facts need be learned—and this is not only a blessing to weak pupils but also a reduction in tiresome drill work.

$$\begin{array}{cccc} 6 & {}^{1}2 & {}^{1}5 & \\ 7 & \not{3} & \not{6} & {}^{1}4 \\ 2 & 8 & 9 & 7 \\ \hline \end{array}$$

How effective the method is can be seen only in application and practice and in the results of the classes and controlled experiments. Teachers long addicted to their own methods may find it hard to adapt themselves to something new and different—especially in the subtraction of whole numbers. Yet, the more they'll use it the easier and the sooner it will come to them. Such has been my experience.

EDITOR'S NOTE. Mr. Mossel's proposal for using complements and "raising" subtrahend digits will, in most cases, result in neater appearing work and he claims added ease and greater accuracy. His argument will be well taken by those many adults who have difficulty in subtraction where a "change" in digits in whole numbers, fractions, and measures must be made. The method suggested requires some understanding of numbers and a little ability to think and that is good. It may seem a little cumbersome in an exercise such as $8\frac{1}{4}-3\frac{3}{4}$. Also, some teachers may wish to do the work with whole numbers a bit differently than has been proposed. We want understanding, ease of operation, and above all accuracy of results. Let us not be afraid to experiment but let us also know what we are doing as we experiment. We want children to understand and to learn arithmetic.

More About Subtraction

A Procedure for Understanding

While reading John A. Sestanovich's "But the Teacher Didn't Show Us that Way," THE ARITHMETIC TEACHER, February, 1959. I was struck with my own problem. Elementary teachers are perennially faced with the problem of borrowing. It seems to be an ages old dilemma of students. After probing in the dark for several years I finally hit upon an idea that seems to have solved the problem. I present it here with the sincere hope that it will be examined and the appropriateness of this approach be evaluated.

I finally realized that far too many children from the sixth grade down do not fully grasp the idea of borrowing from something they already have. These children do not fully comprehend place value. Rather, they see all numbers as a whole. Seven and seven tens are not different by reason of positional value, they are different only because seventy is more than seven. Although we try over and over again to establish this analytical approach (and finally succeed), I had to face up to the fact that many young children fail to grasp this understanding completely. It is only a partial understanding or no understanding. They do not grasp the idea of ones, tens, hundreds, etc. and fully comprehend it. They memorize and can recite the place value columns but they find this of little use. I therefore asked myself, "why teach borrowing? Why not teach so as to provide a potential for this understanding

when the concept of place value falls into place?"

With these questions in mind, I began searching for a subtraction method that would meet the demands of little or no understanding of positional value (while at the same time the method would be within the confines of this concept) to be used until each child began to grasp the significance of place value.

About a year ago I found an approach to subtraction that seems to have immediate contact with a child's reality while at the same time it encourages the child toward a final and complete understanding of borrowing and place value. It is an approach that leads toward discovering what borrowing means. The following approach was finally decided upon:

1. Keep working on place value.
2. For a while at least, go along with the idea that if one has 34 items from which 16 of the items are subtracted, one does not borrow; rather, the 16 items are discarded one by one until all 16 are gone, the result being 18 items.
3. We began to subtract like this. In the example 34 − 16 we have:

$$\begin{array}{r} 34 \\ -16 \\ \hline \end{array}$$

First, subtract the six from the thirty-four and we have twenty-eight with the two tens in the place of three tens—the three tens being crossed out and the eight ones in the answer because they have been subtracted while the tens have not. We have a problem that looks like this:

$$\begin{array}{r} {}^{2}\not{3}4 \\ -16 \\ \hline 8 \end{array}$$

Next, subtract the tens.

$$\begin{array}{r} {}^{2}\not{3}4 \\ -16 \\ \hline 18 \end{array}$$

And the problem is concluded.

For those who cannot understand the borrowing concept when the zero is to be dealt with this is a tremendous approach. Take this example.

$$\begin{array}{r} 3{,}000 \\ -458 \\ \hline \end{array}$$

Think three thousand less eight equals 2,992 and we have:

$$\begin{array}{rrrr} {}^{2}\not{3} & {}^{9}\not{0} & {}^{9}\not{0} & 0 \\ - & 4 & 5 & 8 \\ \hline & & & 2 \end{array}$$

Think 2990 less 50 and we have: 2940

$$\begin{array}{rrrr} {}^{2}\not{3} & {}^{9}\not{0} & {}^{9}\not{0} & 0 \\ - & 4 & 5 & 8 \\ \hline & & 4 & 2 \end{array}$$

Think 2900 less 400 and we have: 2500

$$\begin{array}{rrrr} {}^{2}\not{3} & {}^{9}\not{0} & {}^{9}\not{0} & 0 \\ - & 4 & 5 & 8 \\ \hline & 5 & 4 & 2 \end{array}$$

Think 2000 less nothing and we have: 2000

$$\begin{array}{rrrr} {}^{2}\not{3} & {}^{9}\not{0} & {}^{9}\not{0} & 0 \\ - & 4 & 5 & 8 \\ \hline 2 & 5 & 4 & 2 \end{array}$$

Here then is one teacher's (sixth grade) approach to borrowing that seems to lead toward discovering and understanding borrowing when and if the child reaches this stage of development.

Contributed by
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Reviews of Books and Materials

Edited by E. GLENADINE GIBB

Man and Number, Donald Smeltzer. New York: Emerson Books, Inc., 1958. Cloth, viii+114 pp., $2.50.

By implication this short book about the history of number is of eminent value to anyone concerned with the teaching of elementary mathematics. By implication only, since the author—although fully aware of its significance for teaching—does not aim to be explicit about its application to education. However, the author's concise and continuous manner of presentation, the clarity in language and thought, and his final elucidation of the concept of number, make this volume a true guidebook for teachers and student-teachers alike. Whether he discusses finger symbols, number words, number systems, or number recording, he never loses sight of the main trend in the evolution of number, of the integration between these various topics from which the present concept of number emerges. With his comparison of "tally systems" and "code systems," his exploration of the idea of "collective units," his scrutiny of calculating devices, to give but a few examples, he proceeds along historical as well as psychological lines.

There have been other books available on the same topic, but none with such direct bearing on "developmental" teaching. Yet the very excellence of this book lets one regret that the author left all educational inferences up to his readers. Some highlights of the book may be easily missed since their potential impact on teaching is not specifically pointed out. The most important, it seems to this reviewer, is the author's pithy statement about the difference between tally symbols and numerals: "We may say that

$$|||+||||=\begin{matrix}|||| \\ |||\end{matrix}$$

symbolizes an operation based on counting, while $3+4=7$ symbolizes a relationship between numbers." For many years now teachers have had to follow a faulty philosophy in regard to the teaching of the so-called "combinations," illustrating relationships between numbers by moving or pushing groups of objects together or apart. The adequate symbols for such activities with objects are tally marks, not numerals in their fixed relation, which can be appropriately illustrated on the number line. Although pupils are taught counting and grouping of objects in the first grade, they are put through the same paces in the next grade, and thus are held back on the same primitive level. The distinction made by Smeltzer suggests that another approach to the "combinations" is in order.

In a book of this size one should perhaps not expect an extensive bibliography. But it is unusual that a work of European scholarship should make no reference to technical literature in any language other than English. Moreover, the list of books and articles given is not entirely up-to-date, and some of the statements, therefore, are not in keeping with the latest research. The well-supported hypothesis, for example, that the Greek astronomers completed the Babylonian sexagesimal system by adding a 0 for zero—which the Hindus adopted from the Greeks—is not taken into account. (See: B. L. Van Der Waerden, *Science Awakening*, Groningen, 1954.) Recent articles in the *Journal of Classical Philology* have shown that computation with Roman numerals, even in multiplication, is in no way as cumbersome as the author, following tradition, assumes. Literature is available on the specific tribe —the Basutos—who have used the finger tallying system; Smeltzer, though he presents this system well, makes no reference to

them. Nor is the present extensive literature on the derivation of number words always taken into consideration.

However, some shortcomings of detail do not detract from the admirable contribution Smeltzer's book has to make to a real improvement in the teaching of number.

ANITA P. RIESS

Abacus (One of Man's First Counting Machines), Marguerite Brydegaard. Avis Films, P.O. Box 643, Burbank, California; $115.

This film is intended for use in grades 3–6 to teach the concept of place value, and an understanding of the processes of carrying and borrowing through the use of the abacus.

The first part of the film is devoted to man's attempts to find a way of telling "how many" and some of the frustrations involved when he runs out of fingers and toes in an attempt to gain a one-to-one correspondence with these and the sheep on the hillside. A happy solution to his problem leads to the idea of matching one counter to each ten sheep and the eventual development of the abacus.

The abacus used is the open-end type, which does not show the necessity for substituting one ten for ten ones because man has not run out of fingers (space on the ones stick). It does not seem necessary to substitute for the ten counters on the ones stick, so long as there is space on the ones stick to put more counters. However, this type abacus is necessary if the aim is to lead to the method of borrowing which involves the changing of one ten and two ones in the minuend to twelve ones. An alternate suggestion for handling the situation might be to develop the idea of the necessity for place value through the use of the closed abacus and change to the open type for borrowing and carrying. These two concepts can be taught with the closed abacus, but the method involved does not so easily lead to the type of manipulation that is currently in use in many textbooks.

The film is accompanied by an abacus so that the pupils may look at the film and then actually do the manipulations portrayed.

The script contains some expressions such as "When we add we increase" and "When we subtract we decrease" which are at variance with ideas expressed by some authorities that the operations are merely processes of regrouping. Also, one might question the clarity of such statements as "10 times as small."

It would seem to the reviewer that a teacher who is well versed in the principles of the abacus could teach the concepts just as well without the film. However, in a case where the teacher would like some direction the film would be of value.

INA MAE SILVEY

Geometry Can Be Fun, Louis Grant Brandes (Portland, Maine: J. Weston Walch, 1958). Paper, iv+249 pp., $2.50 (teacher edition).

This is a most interesting collection of materials suitable for enriching a class discussion, for programs in mathematics clubs, and for private reading by a pupil. The materials are not all geometric and many items are useful as low as grade six. Here in one book will be found many of the things which usually do not appear in a textbook but which a teacher collects and has available to enliven and stimulate interest in mathematics. Mr. Brandes has collected most of the items from teachers and pupils.

The book is produced by offset printing from typescript on pages 8½ by 11 inches. It is interestingly illustrated with pen-and-ink drawings. The content includes tricks, puzzles, and illusions; stories and biographies; games and jokes; and projects, constructions, and problems. The presentation of the topics is not only interesting but is made attractive by the neatly designed sketches. Teachers will welcome the full classified index, the answers, and the well-chosen bibliography.

This reviewer recommends the book as one that should be available for all teachers of mathematics beginning with grade six.

BEN A. SUELTZ

Understanding and Teaching Arithmetic, E. T. McSwain, Ralph J. Cooke, New York: Henry Holt and Company, 1958. Cloth, xi+420 pp., $5.50.

The publication of this volume is without a doubt the result of a great deal of intensive analysis of the field of arithmetic and an exploration into the underlying meanings of the concepts involved. The book contains many sections which should prove to be extremely helpful to a teacher who is intent on taking the subject out of the area of rote or mechanical learning. These features include:

(1) exploratory questions which focus one's thinking on crucial issues
(2) diagnostic questions which might well be used to determine areas where reteaching is necessary
(3) practice and self-evaluation exercises for the student preparing for teaching
(4) suggested, purposeful activities suitable as projects for teacher education classes
(5) concise summary statements at chapter endings
(6) well chosen references at chapter endings for use in collateral reading

The over-all content covers the usual topics found at all grade levels, with a somewhat less intensive treatment of applications in the fields of business, industry, and measurement. Concluding chapters give particular emphasis to problem solving, evaluation, and curriculum considerations such as basic philosophy, sequence, and meeting the needs of the gifted children.

The reader of this volume cannot help but be impressed with its paramount objective—to expose the pupil to mathematical meanings. Some of the presentations are common property of most similar volumes. Unique to this treatment, however, is an extreme extension of the implications of place value, particularly in the processes of multiplication and division, and much more especially with the involvement of decimals. Illustrative is the following teaching example in multiplication, with the accompanying explanation (page 83):

H	T	O
	2	0
	3	0
		0
		0
		0
6	0	
6	0	0

"In example (a), 30×20, the question asked and to be answered is, "What number equals 30 (3 tens) groups of 2 tens each or 3 tens×2 tens=?" By thinking of 20 as 2 tens and 0 ones and applying the principle of computing with like units, first ones and then tens, the steps are: Step 1—think, "The multiplier consists of 3 tens and 0 ones. First multiply by the ones. There are no ones in the multiplier. Write 0 in the ones place in the first partial product." Step 2—think, "Now multiply by the tens. There are no ones to multiply by 3 tens. Write 0 in the tens place in the third partial product." Step 3— think, "What number equals 30 (3 tens) groups of 2 tens each?" The answer is 60 tens. Write 6 in the hundreds place in the fourth partial product. The total product in example (a) is the sum of 6 hundreds, 0 tens, and 0 ones or 600."

The following illustration is presented, among several others, as a preferred method of teaching the multiplication of decimals:

.246	
×38.	
.048	8 ones×.006
.32	8 ones×.04
1.6	8 ones×.2
.18	3 tens×.006
1.2	3 tens×.04
6.	3 tens×.2
9.348	Product

Analogous procedures requiring though processes involving decimals at each step, are suggested for examples in division of decimals. Space does not permit giving additional illustrations.

It is the writer's feeling that the procedures described above as well as others presented elsewhere in the volume, are too complex for the average child, and that the insistence that meanings be extended to this point might well give occasion for errors not encountered in methods as ordinarily taught.

Perhaps the greatest contribution that this volume can make is to challenge the teacher to become more alert to the importance of meanings. General acceptance of some of the teaching suggestions might well be deferred, at least until they have been tried out empirically with children of all levels of ability, or until their efficacy has been established through carefully conducted controlled experiments.

There are a few evidences that the original manuscript was not subjected to critical appraisal before publication. For example, the following arithmetic operation appears as a partial explanation of a division examples in decimals.

$$2\times.004=.008+.25\times.004=$$

$$.008+.001=.009$$

In another place, the presentation of the determination of the formula for the area of a trapezoid requires an understanding of algebraic processes not normally comprehended in the upper elementary grades. The writer feels that other more suitable methods should have been presented.

HARRY C. JOHNSON

Meeting Individual Differences, The Casis School Faculty (Directed and edited by Frances Flournoy and Henry J. Otto), University of Texas Press (Laboratory Schools Publication No. 11) 1959. Paper, v+184 pp., $2.00.

The Casis School Faculty has worked out a splendid monograph which explores the possibilities of individual differences. These include variations in learning time, arithmetical content, methods and materials within certain types of classroom organization. They have described procedures used in the school to meet these individual differences.

In certain chapters, the class worked as a whole, then were grouped for further study according to abilities. In others, the class was gathered as a unit for review in relation to the new concept. To meet this new idea, they were then divided according to readiness—a valuable method! This accelerated the learning. In a third instance, the developmental procedure used was excellent. After the introduction of the new concept, individuals were allowed to work at their own rate of speed.

The reviewers were particularly impressed by the experiment in Chapter Eight, "Third Graders Learn Relationships between Liquid Measures." The group, as a whole, met to discuss the problems involved. Following this, five heterogeneous groups worked with materials to discover liquid measurement relationships, teacher participating. Finally, the whole class met to accumulate learnings. Throughout the study, correlation with other subjects in the curriculum was stressed.

Our criticisms are not many, but we feel justified in mentioning the following:

1. Expensive material for costumes.
2. Even distribution of teacher time among groups of varying sizes—especially a very small group and a very large group.
3. Too much sub-grouping.

In the final chapter, "In Summary and Looking Forward," a consideration of "Above Average Achievers" and "Below Average Achievers," as to time, content, methods and materials has been made in a very clear-cut manner. This is followed by an "Analysis of Procedures Used," giving a concise picture of methods. Finally, under "Issues to be Studied," a list of six problems has been outlined for further study.

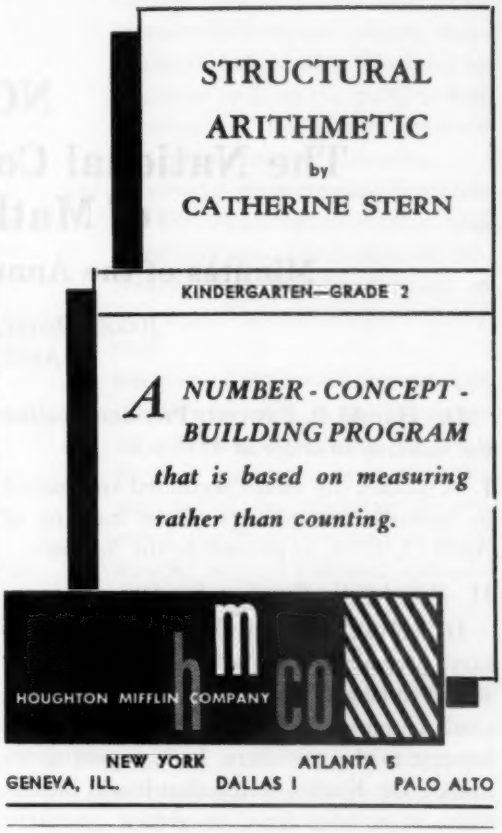
GERTRUDE P. BREWSTER
DORIS B. HOLTHAM

The Structure of Arithmetic and Algebra, May Hickey Maria. New York: John Wiley and Sons, Inc., 1958. Cloth, xiv+294 pp., $5.90.

The author states the aim of this book is to "make available to the non-science student and to the teacher of secondary school mathematics the fundamental concepts that underlie the structure of algebra and arithmetic." Actually, the book does more and does it very well. It could easily be used as a text or for self-study, not only for non-science students and teachers of secondary school mathematics, but also for elementary school teachers, particularly those in the upper elementary grades and in junior high school. Elementary teachers would probably restrict themselves to Chapters 1–7 and 15.

The totality of real numbers is taken as logical entities. Then the main properties of the real numbers are investigated as logical consequences of a system of fundamental assumptions. There is no attempt to start with a minimal set of assumptions, but a sufficiently extensive system is chosen to assure a complete characterization of the totality of real numbers.

The outstanding feature of the book is its logical development. It assumes little or no background, but does demand an inquiring mind and the willingness to do some work. In the beginning chapters, to help those readers with very little experience with algebraic proof, the author analyzes precisely what assertions are made by an axiom and what the axiom does not say. Examples using these definitions and axioms are then worked out in detail with reasons given for each step of a proof. (An alphabetical code of references is used, which, although somewhat cumbersome, makes the detailed examples possible.) Care has been taken to eliminate misunderstanding as, for example, by using a negative sign as a superscript to avoid confusion between the sign of the number and the operational symbol for subtraction.



The exercise lists, although adequate, could be improved by including some numerical problems in more of the lists.

In Chapter 1, sections 1.8, 1.9, and 1.11 might falsely lead an inexperienced reader to believe that background in logic is necessary to understand the logical development. However, this is not the case and concepts not completely clear at first reading should not keep anyone from beginning Chapter 2.

In summary, I feel this is an excellent, though not a popular, treatment of the real number system. Any teacher, elementary or high school, would benefit greatly by reading this book. Indeed, it would also serve as an excellent book for superior high school students by offering insight on the real number system as well as developing skill in abstract algebraic proof.

ROBERT J. TROYER

NCTM
The National Council of Teachers of Mathematics
Minutes of the Annual Business Meeting

BAKER HOTEL, DALLAS, TEXAS
April 3, 1959

Dr. Harold P. Fawcett, President, called the meeting to order at 4:10 P.M.

I. A motion was made, seconded and passed to approve the minutes of the meeting of April 11, 1958, as printed in the *Journals*.

II. *Report of the Recording Secretary*

Dr. Houston T. Karnes, Recording Secretary, gave a brief review of the actions of the Board during the past year. This review included items which were thought to be of interest to the members. In his opening remarks, Dr. Karnes stated that it was impossible, in a brief time, to give a complete summary of the Board meetings. He pointed out that a large share of the meetings are devoted to committee reports and then discussing matters before decisions are reached. He exhibited a folder of the committee reports which were submitted at the Board meeting immediately preceding the Dallas convention, and indicated that the folder of reports would form a book of considerable size. The review follows:

A. Immediately following the Cleveland meeting, the Board authorized the printing of a new Directory. This, of course, has been done and is a distinct aid to many people.

B. At the New York meeting, Dr. Robert E. Pingry, of the University of Illinois, was elected to a three-year term as Editor of *The Mathematics Teacher* to succeed Dr. Henry Van Engen, who has served in this capacity for six years—the maximum time allowed under our Bylaws.

C. Dr. Irvin Brune has been authorized to supervise the writing of a Guidance pamphlet to replace the one now in existence.

D. Future Meetings of the Council:

	Spring	*Summer*	*Christmas*	*Joint*
1959		Ann Arbor, Michigan		
1960	Buffalo	Salt Lake City, Utah	Tempe, Arizona	
1961	Chicago	Toronto, Canada		
1962	San Francisco	Madison, Wisconsin		
1963	Pittsburgh	Boulder, Colorado		

In addition, the Board approved a one-half day meeting in conjunction with the NEA St. Louis meeting this summer. Also, a full day meeting in conjunction with the Chicago meeting of the M.A. of A. next January. This gives us four meetings for 1960.

The Board decided to schedule no future Christmas meetings as a regular order, but to consider any invitations which might be extended.

E. Future Yearbooks:
(1) The 25th Yearbook, *Instruction in Arithmetic*, will be released in the Spring of 1960;
(2) The 26th Yearbook, *The Evaluation of Achievement in Mathematics*, will be released in the Spring of 1961.

F. In view of increased costs of a convention, the Board has increased the regis-

tration fee for members from fifty cents to $1.00, and for non-members from $1.00 to $3.00. This change will become effective with the 1960 Buffalo meeting.

G. The anticipated report of the Secondary School Curriculum Committee will appear in the May issue of *The Mathematics Teacher*. Reprints will be available.

H. The Board has increased the number of issues of THE ARITHMETIC TEACHER from six to eight per year.

I. In view of the steadily increasing duties of the President, the Board has given this office much relief by releasing the President from Convention Program responsibilities. For all future meetings, the program will be the responsibility of a program committee, composed of the four vice-presidents.

J. *Budget:* Time does not permit a detailed report of the new budget at this session. The budget for 1959–60, as adopted by the Board, will, however, be published in the October issue of our *Journals*. It is thought well to report on three items today.

(1) The operation of the Council is steadily growing and has reached such proportions that the Board has approved an operational budget of approximately $195,000 for 1959–60. We are thankful that the income permits this to be done. This means increased and better service to the membership and increased activities in the field of mathematics education. It should be said that the budget is not set at this figure because the money is available through an increasing membership, but rather an increasing membership and increasing demands due to the present impetus on mathematics makes a budget of this size necessary. If the money was not available, through dues, it would have to be transferred from our reserve security and thus deplete this fund in one year or from the publications account and thus greatly decrease our effectiveness in this area.

(2) In view of the increasing demands on the Washington Office and especially upon the Executive Secretary, it is found necessary to add a professional assistant to our staff.

(3) Because of the increasing amount of editorial work in the Washington Office and because the NEA finds it necessary to curtail the services of its editorial department, the Board deemed it necessary to add an editorial assistant to the Washington Office.

III. *Report of the Executive Secretary*

Mr. M. H. Ahrendt, Executive Secretary, gave the following brief report.

A. The present staff of the Washington Office consists of the Executive Secretary and twelve secretarial and clerical persons. In December, the Office was moved to larger quarters in the new NEA Building. Mr. Ahrendt indicated that these quarters would no doubt be somewhat crowded by this coming fall in view of the manner in which the work was growing.

B. On February 1, 1959, the total membership and subscriptions was 22,215. This was an increase of 4,000 over the same date of the previous year. At the present rate, this total should reach 23,500 or 24,000 by May 1. When most organizations increase dues, there is usually a decline in membership. This, however, did not happen in the case of the Council.

C. In order to further show the growth of the Council, Mr. Ahrendt gave the printing orders for the April, 1959 issue of our *Journals*. The printing order for *The Mathematics Teacher* was 20,500,

for THE ARITHMETIC TEACHER, 9,500 and for *The Mathematics Student Journal*, 80,000.

D. Mr. Ahrendt reported that the financial condition of the Council was in excellent condition, with income ahead of budgetary estimates. A complete financial statement will appear in the October issue of the *Journals*.

E. Mr. Ahrendt stated that the addition of the addressograph machinery was not only a help in the Washington Office, but also as an aid to other professional organizations which had purchased the services of this equipment. To date, the machinery has been used for all or part of the list for thirty-one separate mailings.

F. In addition to the membership listing in the Washington Office, a new registry has been developed. This is The U. S. Registry of Mathematics and Science Teachers. This registry was developed in cooperation with the National Science Teachers Association and subsidized by the National Science Foundation. At the present, it contains 120,000 names of which 80,000 teach at least one course in mathematics.

G. New Publications. In addition to the *24th Yearbook* and several supplementary publications, there are three publications which should be of interest to the teachers of mathematics. They are:

(1) Career Pamphlet. This pamphlet was produced with the cooperation of NSF-NRC. All members of the Council should receive a copy of this publication before too long.

(2) Articles by George Boehm in *Fortune Magazine*. Reprints of these articles are being assembled and all members of the Council should receive a copy within a short time.

(3) *Mathematics for the Academically Talented Student*. This publication was developed in cooperation with the NEA project on the academically talented student. Copies of this publication are available in the Washington Office.

IV. *Report of the President*

President Fawcett stated that a report of his activities and the over-all work of the Council would extend this session much beyond its scheduled closing time. He stated further that the reports being given by others at this meeting would give the membership a good over-all picture of the work of the Council during the past year.

President Fawcett said that, under the Bylaws, the President has two duties. They are:

(1) The usual responsibilities associated with the office of President, and

(2) To preside at the Board Meetings.

He remarked that it did not take him long to find that these responsibilities were great and time consuming. He thanked the membership for the opportunity that had been given him, and said that he had developed two new respects for the Council.

(1) The effect that the Council had had upon his own life, and

(2) The respect with which the Council is held in professional circles.

The work of the Council, President Fawcett stated, was carried on through an overwhelming committee structure. At the present, there are some twenty-eight committees, containing a personnel of 139 people. The work of these committees delves into all phases of mathematics education. From the work of these committees, important developments are taking place, which will be of great aid to the teachers of mathematics.

President Fawcett extended his thanks and appreciation in general to the membership for the excellent cooperation which he had received during the past year, and more specifically to the work of the members of the Board, the editors and to the members of the various committees. In addition, he expressed his thanks and appreciation to the four vice-presidents and to former President Mrs. Marie S. Wilcox, for the excellent work

they did in assisting with developing the programs for the meetings of the Council.

V. *Problem-Policy Committee*

This past winter, on the recommendation of the President, the Board authorized the establishment of a new committee to be known as the Problem-Policy Committee. In introducing the chairman of this committee, President Fawcett made the following statement:

According to the Bylaws adopted March 29, 1949 and amended April 24, 1954, the purpose of the Council is "to assist in promoting the interests of mathematics in America, especially in the elementary and secondary fields." Since its creation thirty-nine years ago, it has steadfastly served this high purpose and throughout this period has continued to emphasize the importance of mathematics in a program of American education. Until recently it has, in fact, been the only professional organization dedicated to the improvement of mathematics teaching on both the elementary and secondary levels. Within recent years, however, there has been a decided change in the intellectual and academic climate of America, and today the improvement of mathematics education from kindergarten through college is a problem which has commanded the active interest of other professional organizations and is of concern to all thoughtful people. Significant curricular studies are in process. New instructional materials are being developed. An in-service program of large proportions is now operating and the collegiate program for the mathematics teachers of tomorrow is under revision. Large financial assistance is available for all of the projects, and it is evident that the original purpose of the Council, "to assist in promoting the interest of mathematics in America, especially in the elementary and secondary fields," is now receiving widespread support. What then should be the role of the Council in this changed intellectual climate?

To study this and other policy problems, I have created, with the approval of the Board of Directors, a "Problem-Policy Committee" with responsibility for keeping alert to movements in mathematics and in mathematics education, both at home and abroad, of defining areas of genuine concern to the Council and of proposing to the Board those policies which will best serve the interest of mathematics education. The committee, consisting of Phillip Jones, John Mayor, Philip Peak and Howard Fehr, Chairman, has had one meeting, and at this time it is appropriate to have a preliminary report from the chairman, Howard Fehr.

VI. *Report of the Problem-Policy Committee*

Dr. Fehr gave his report verbally using the following outline.

A. *Origin: Events Occurring During the Last Six Years*
   (1) Upsurgence of international need for mathematics. Impetus in mathematics education.
   (2) Formational support of other groups.
   (3) Relation to other groups in our services to the membershlp of the Council.

B. *What are the problems of the Council and what should be its policy?*
   (1) Teacher education: regional conferences.
   (2) Publications: monographs for teachers—yearbooks for parents and laymen.
   (3) Annual conferences.
   (4) Adult education program.
   (5) Teacher certification and teaching standards.
   (6) Role of television.
   (7) Elementary school.

C. As a result of the deliberation on the topics under A and B above, the Committe recommended that a meeting be held as early as possible this fall, composed of some twenty-odd participants, to deliberate upon the future work of the Council in view of the wide range of activity in the field of mathematics today. The Board adopted this recommendation and authorized the President to proceed with plans for the meeting. From the results of the meeting, it is expected that a well defined program and policy would be adopted for the work of the Council during the next several years.

VII. *Report of the 1959 Committee on Nominations and Elections*

Dr. Milton W. Beckmann, chairman, gave the report for the committee. He stated that, in addition to a meeting of the Committee in Cleveland last Spring, the Committee met for a day and a half in Chicago in the early part of the summer. During this meeting, careful consideration was given to the many recommendations which had been received. As a result of this meeting, the following were nominated:

(1) Vice-President for Colleges
Phillip S. Jones*
Z. L. Loflin
(2) Vice-President for Junior High Schools
Marian C. Cliffe
Mildred B. Cole*
(3) Directors
Carol V. McCammon
Philip Peak*
Christine Poindexter
Irene Sauble
Oscar F. Schaaf*
Henry Van Engen*

Those names above marked by an asterisk are the persons who were duly elected.

Dr. Beckmann stated that a total of 3,747 votes were cast. He went on to say that it was regrettable that such a small percentage of the 15,000 odd membership took part in the voting. In closing, he suggested that more interest should be generated in the election of officers.

VIII. *Report of the Resolutions Committee*

The Resolutions Committee was composed of Miss Alice M. Hach and Dr. H. Vernon Price. Miss Hach presented the following report.

The National Council of Teachers of Mathematics depends upon loyal members and supporters for its existence. Many people devote a great deal of energy and time in making a convention a success. It is recognized that a large part of this effort is contributed by the local committees. It is proper, therefore, that the National Council express its appreciation to the Greater Dallas Council of Teachers of Mathematics and the Dallas Independent School District.

Acting as host for a convention of this size has been a major undertaking. But you in Texas have been noted for doing things on a grand scale. Your hospitality and your contribution to the success of this 37th annual meeting have been in keeping with this tradition.

Following the reading of this resolution, Miss Hach moved its adoption. The motion was duly seconded and passed unanimously.

IX. Dr. H. C. Christofferson, Former President, moved that the President and the Board be extended a vote of thanks and complimented highly on the outstanding work which was accomplished during the past year. This motion was duly seconded and passed unanimously.

X. Dr. Donovan A. Johnson moved that Dr. Henry Van Engen, retiring editor of *The Mathematics Teacher*, be given a vote of thanks and complimented for the outstanding work he did during the past six years as editor of this *Journal*. The motion was duly seconded and passed unanimously.

XI. There being no further business, the meeting was duly adjourned at 5:10 P.M.

Respectfully submitted
HOUSTON T. KARNES
Recording Secretary

MATHEMATICS TESTS AVAILABLE IN THE UNITED STATES

by Sheldon S. Myers

A listing, as complete as possible, of all the mathematics tests available in the United States. Gives information as follows: name of test, author, grade levels and forms, availability of norms, publisher, and reference in which review of test can be found.

16 pp. 50¢ each.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W.
Washington 6, D.C.

Annual Financial Report

Below is a brief financial report of the Council for the fiscal year ending May 31, 1959. The reader will note that from a financial viewpoint the year was a good one for the Council.

No doubt the most striking feature of the business of the past year was the continued rapid increase in membership in spite of an increase in dues. Nearly all professional organizations have experienced a temporary decline in membership or in membership growth after an increase in membership dues. By contrast the membership of the Council increased at the greatest rate in its history.

Another noteworthy development was an increase of about 50 per cent in the sale of publications over the sales of the previous year. These two factors produced unexpected income which has greatly strengthened financially the Council.

RECEIPTS AND EXPENDITURES
OF THE
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
FISCAL YEAR, JUNE 1, 1958–MAY 31, 1959

June 1, 1958 Total Cash Resources		$53,345.08
RECEIPTS		
Memberships with *Mathematics Teacher* subscriptions	$ 72,258.73	
Memberships with ARITHMETIC TEACHER subscriptions	23,351.48	
Institutional subscriptions to *Mathematics Teacher*	27,147.82	
Institutional subscriptions to ARITHMETIC TEACHER	20,715.29	
Subscriptions to *Mathematics Student Journal*	18,174.94	
Sale of advertising space in *Mathematics Teacher*	7,260.08	
Sale of advertising space in ARITHMETIC TEACHER	1,473.07	
Rental of membership list	2,520.31	
Interest on U. S. Treasury Bonds	375.00	
Net profit from conventions	3,292.58	
Miscellaneous	461.66	
Sale of publications		
Yearbooks	35,828.92	
Supplementary publications, etc.	24,198.57	
Total receipts	$237,058.45	
EXPENDITURES		
Washington office	$ 65,219.31	
Purchase of Addressograph equipment	5,203.43	
President's office	2,849.35	
Vice-presidents' office expenses	125.00	
Mathematics Teacher	38,083.65	
ARITHMETIC TEACHER	14,083.50	
Mathematics Student Journal	12,179.75	
Secondary School Curriculum Committee	5,432.02	
Elementary School Curriculum Committee	898.58	
Committee work	3,194.14	
Travel by Board members	4,413.56	
Preparation and printing of yearbooks	20,691.53	
Preparation and printing of supplementary publications	8,202.53	
Distributing special publications	3,724.54	
Printing and mailing *Membership Directory*	5,669.30	
Storage and shipment of publications, miscellaneous	853.98	
Total expenditures	$190,824.17	
INCREASE IN CASH RESOURCES		46,234.28
May 31, 1959—Total Cash Resources		$99,579.36

National Council of Teachers of Mathematics
Operating Committees
1959-1960

Secretary of the Board

Houston T. Karnes, Baton Rouge, La. (1960)

National Council Representatives

A.A.A.S. Cooperative Committee on Science and Mathematics

Bruce Meserve, Montclair, N. J. (1960)

Educational Advisory Committee to Science Service

Veryl Schult, Washington, D. C. (1960)
Helen Cooper, Bethesda, Md. (1961)

Conference Board of the Mathematical Sciences

Burton Jones, Boulder, Colo. (1961)
John Mayor, Washington, D. C. (1960)

United States Commission on Mathematics Instruction

Howard F. Fehr, New York, N. Y. (1960)
Eugene Northrop, Chicago, Ill (1961)

A.A.A.S. Council

Phillip Jones, Ann Arbor, Mich. (1961)
Veryl Schult, Washington, D. C. (1961)

Committees

1. *Executive Committee* (1960)
   Harold Fawcett, Columbus, O.
   Phillip Jones, Ann Arbor, Mich.
   Philip Peak, Bloomington, Ind.
2. *Problem-Policy Committee* (1960)
   Howard F. Fehr, New York, N. Y., Chairman
   John R. Mayor, Washington, D. C.
   Phillip Jones, Ann Arbor, Mich.
   Philip Peak, Bloomington, Ind.
3. *Budget Committee*
   Houston T. Karnes, Baton Rouge, La., Chairman (1960)
   Bruce Meserve, Montclair, N. J. (1962)
   Vernon Price, Iowa City, Iowa (1961)
   Duties: To prepare the operating budget of the Council for the year 1960–1961 and to present it to the Board at its annual meeting in Buffalo.
4. *Nominations and Elections* (1960)
   Ida Bernhard Puett, Atlanta, Ga., Chairman
   Milton Beckmann, Lincoln, Neb.
   Charles Butler, Kalamazoo, Mich.
   Howard F. Fehr, New York, N. Y.
   Kenneth Henderson, Urbana, Ill.
   Houston T. Karnes, Baton Rouge, La.
   Albert Linton, Philadelphia, Pa.
   Ann Peters, Keene, N. H.
   Oscar Schaaf, Eugene, Ore.
   Duties: To provide for the spring election in 1960 a slate of nominees all of whom have indicated their willingness to serve if elected.
5. *Nominations and Elections* (1961)
   Oscar Schaaf, Eugene, Ore., Chairman
   Harold Fawcett, Columbus, O.
   W. Eugene Ferguson, Newton, Mass.
   Lenore John, Chicago, Ill.
   Houston T. Karnes, Baton Rouge, La.
   W. C. Lowry, Charlottesville, Va.
   Ida Bernhard Puett, Atlanta, Ga.
   Max Sobel, Fair Lawn, N. J.
   Lottchen Hunter, Wichita, Kan.
   Duties: To provide for the spring election in 1961 a slate of nominees all of whom have indicated their willingness to serve if elected.
6. *Nomination of Editor for* THE ARITHMETIC TEACHER (1960)
   Maurice Hartung, Chicago, Ill., Chairman
   Irene Sauble, Detroit, Mich.
   Ben Sueltz, Cortland, N. Y.
   Duties: To nominate two persons, each of whom has indicated his willingness to serve as editor of THE ARITHMETIC TEACHER if elected by the Board, and to present these nominees to the Board at its 1959 summer meeting at Ann Arbor.
7. *Reporting Elections* (1960)
   Ida Bernhard Puett, Atlanta, Ga., Chairman
   Myrl Ahrendt, Washington, D. C.
   Harold Fawcett, Columbus, O.
   Duties: To certify the results of the annual election to membership on the Board of Directors and to report these results at the annual business meeting.
8. *Affiliated Groups*
   Eugene Smith, Wilmington, Del., Chairman (1961)
   Northeastern: Catherine Lyons, Pittsburgh, Pa. (1960)
   North Central: Virginia Pratt, Omaha, Neb. (1961)
   Western: Kenneth Skeen, Concord, Cal. (1961)
   Southeastern: Houston Banks, Nashville, Tenn. (1960)
   Central: Adeline A. Riefling, St. Louis, Mo. (1961)
   Southwestern: Eunice Lewis, Norman, Okla. (1961)
9. *Membership* (1960)
   Mary Rogers, Westfield, N. J., Chairman
   Myrl Ahrendt, Washington, D. C.
   Marian C. Cliffe, Glendale, Cal.
   Mary Rickey, Cedar Rapids, Iowa
   Eugene Smith, Wilmington, Del.
   Duties: To suggest ways and means of increasing membership in the Council and to translate these suggestions into action through cooperative work with the Affiliated Groups and the State Representatives.
10. *Place of Meeting*
    Alice Hach, Racine, Wis., Chairman (1960)
    Mabel Baker, Pittsburgh, Pa. (1961)
    Forest N. Fisch, Greeley, Colo. (1960)

Eugene Nichols, Tallahassee, Fla. (1961)
James Nudelman, Cupertino, Cal. (1961)
Myrl Ahrendt, Washington, D. C. ex officio

Duties: To study, plan and report to the Board concerning geographic and strategic locations of all Council meetings and to recommend to the Board a planned sequence of convention cities through 1966.

11. *Research*
John Kinsella, New York, N. Y., Chairman (1962)
Kenneth Brown, Washington, D. C. (1961)
Howard F. Fehr, New York, N. Y. (1961)
Sheldon Myers, Princeton, N. J. (1962)
Fred Weaver, Boston, Mass. (1961)

Duties: To promote research in mathematics education, to provide a means of summarizing and publishing significant research in this area and to plan the program for a Research Section at the annual meeting.

12. *The Mathematics Teacher* (1960)
Robert Pingry, Urbana, Ill., Editor (1962)
Jackson Adkins, Exeter, N. H.
Mildred Keiffer, Cincinnati, O.
Daniel Lloyd, Washington, D. C.
Z. L. Loflin, Lafayette, La.
Ernest Ranucci, Newark, N. J.

13. THE ARITHMETIC TEACHER (1960)
Ben A. Sueltz, Cortland, N. Y., Editor (1960)
Marguerite Brydegaard, San Diego, Cal.
John R. Clark, New Hope, Pa.
Glenadine Gibb, Cedar Falls, Iowa
Joseph J. Urbancek, Evanston, Ill.

14. *The Mathematics Student Journal* (1960)
W. W. Sawyer, Middletown, Conn., Editor (1961)
Arnold Ross, Notre Dame, Ind.
Oscar Schaaf, Eugene, Ore.

15. *Supplementary Publications*
L. A. Ringenberg, Charleston, Ill., Chairman (1960)
Houston Banks, Nashville, Tenn. (1960)
Marguerite Brydegaard, San Diego, Cal. (1961)
Edwin Eagle, San Diego, Cal. (1960)
Kenneth Henderson, Urbana, Ill. (1962)
Burton Jones, Boulder, Colo. (1961)
Margaret Joseph, Milwaukee, Wis. (1962)
Jesse Osborn, St. Louis, Mo. (1961)
Mildred Keiffer, Cincinnati, O. (1960)
Helen A. Schneider, Oak Park, Ill. (1961)
H. C. Trimble, Cedar Falls, Iowa (1961)
James Ulrich, Arlington Heights, Ill. (1960)

Duties: To plan a program of small publications in addition to the three journals and the Yearbooks, to solicit and evaluate manuscripts received and to submit to the Publications Board those which seem acceptable for publication.

16. *Publications Board*
Clifford Bell, Los Angeles, Calif., Chairman (1960)
Henry Swain, Winnetka, Ill., Chairman (1961)
Henry Van Engen, Madison, Wis., Chairman (1962)
The Mathematics Teacher, Robert Pingry (1962)
THE ARITHMETIC TEACHER, Ben Sueltz (1960)
Mathematics Student Journal, W. W. Sawyer (1961)
Supplementary Publications, L. A. Ringenberg (1960)
Yearbook Planning Committee, Bruce Meserve (1961)

Duties: To propose and coordinate publication policies of the Council, to make recommendations to the Board concerning major publications and to report action on minor publications to the President.

17. *Yearbook Planning*
Bruce Meserve, Montclair, N. J., Chairman (1961)
Houston Banks, Nashville, Tenn. (1962)
Robert Pingry, Urbana, Ill. (1960)

Duties: To report on status of yearbooks in progress, to recommend topics for future yearbooks and to suggest possible editors and committee members.

18. *Yearbooks*
18.1—Twenty-Fifth (Arithmetic)
Foster Grossnickle, Jersey City, N. J., Chairman
Dan Dawson, Stanford, Calif.
Ida Mae Heard, Lafayette, La.
Irene Sauble, Detroit, Mich.
Herbert Spitzer, Iowa City, Iowa
Louis C. Thiele, Detroit, Mich.
18.2—Twenty-Sixth (Evaluation)
Donovan Johnson, Minneapolis, Minn., Chairman
Robert Fouch, Tallahassee, Fla.
Glenadine Gibb, Cedar Falls, Iowa
Robert Pingry, Urbana, Ill.
Max Sobel, Fair Lawn, N. J.
Ben A. Sueltz, Cortland, N. Y.
Fred Weaver, Boston, Mass.

19. *Elementary School Curriculum*
Fred Weaver, Boston, Mass., Chairman (1961)
Joyce Benbrook, Houston, Tex. (1960)
Laura Eads, New York, N. Y. (1960)
Ann Peters, Keene, N. H. (1962)
Irene Sauble, Detroit, Mich. (1961)
Henry Van Engen, Madison, Wis. (1962)

20. *Secondary School Curriculum* (1960)
Frank B. Allen, La Grange, Ill., Chairman
Jackson B. Adkins, Exeter, N. H.
Howard F. Fehr, New York, N. Y.
A. S. Householder, Oak Ridge, Tenn.
Lottchen Hunter, Wichita, Kan.
Burton Jones, Boulder, Colo.
John R. Mayor, Washington, D. C.
Bruce Meserve, Montclair, N. J.
Sheldon Myers, Princeton, N. J.
E. B. Newell, Indianapolis, Ind.
Alfred Putnam, Chicago, Ill.
Elizabeth Roudebush, Seattle, Wash.
Helen Walker, New York, N. Y.
Marie Wilcox, Indianapolis, Ind.
Lynwood Wren, San Fernando, Cal.

Magnus Hestenes, Los Angeles, Cal. (Advisor)

21. *Teacher Education and Certification*
    Henry Syer, Kent, Conn., Chairman (1961)
    Charles Atherton, Shepherdstown, W. Va. (1962)
    Kenneth Brown, Washington, D. C. (1960)
    Daniel Lloyd, Washington, D. C. (1961)
    Robert Kalin, Tallahassee, Fla. (1962)
    David Page, Urbana, Ill. (1960)
    Richard Purdy, San Jose, Cal. (1962)
    Duties: To study the present certification requirements for teachers of mathematics in the respective states including the mathematical background of elementary school teachers and to recommend appropriate policy to the Board of Directors.
22. *Television*
    Joseph Hooten, Tallahassee, Fla., Chairman (1960)
    George Anderson, Millersville, Pa. (1960)
    Emil Berger, St. Paul, Minn. (1961)
    Mary Nesbitt, Miami, Fla. (1962)
    David Wells, Pontiac, Mich. (1960)
    Duties: To study the status of television in the teaching of mathematics and to propose policy to the Board of Directors concerning the use of this medium in mathematics education.
23. *Mathematics for the Talented* (1960)
    Julius H. Hlavaty, New Rochelle, N. Y., Chairman
    Mary Lee Foster, Arkadelphia, Ark.
    Frances Johnson, Oneonta, N. Y.
    Glen Vanatta, Indianapolis, Ind.
    Robert Fouch, Tallahassee, Fla.
    Joseph Payne, Ann Arbor, Mich.
    Harry Ruderman, New York, N. Y.
    Duties: To develop plans and proposals designed to improve and strengthen the mathematics education of talented students.
24. *Cooperation with Industry* (1960)
    Marie Wilcox, Indianapolis, Ind., Chairman
    Jesse Cardwell, Dallas, Tex.
    William Glenn, Pasadena, Cal.
    Kenneth Kidd, Gainesville, Fla.
    Zeke Loflin, Lafayette, La.
    G. A. Rietz, New York, N. Y.
    Lauren Woodby, Mt. Pleasant, Mich.
    Duties: To publish in the official journals of the Council descriptions of cooperation between industry and mathematics education and to plan the program for a "Cooperation with Industry" section at the annual meeting.
25. *International Relations*
    E. H. C. Hildebrandt, Evanston, Ill., Chairman (1962)
    Ernest Ranucci, Newark, N. J. (1961)
    Alfred Putnam, Chicago, Ill. (1961)
    Veryl Schult, Washington, D. C. (1960)
    Duties: To extend the hospitality and services of the Council to mathematics teachers and students from the United States who are visiting other countries and to foreign mathematics teachers and students in this country; to cooperate with international organizations in securing an interchange of information concerning mathematics education, and to plan the program for an "International Section" at the annual meeting.

Clark

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tions that may be answered as the experiment progresses are these:

1. Have you been able to know your children better, to evaluate their abilities and progress better than before?
2. Have you noticed any correlation between ability in reading and ability in arithmetic?
3. Has your curriculum changed to any extent, other than a change in some of the techniques involved in presenting the material?
4. Have you felt more successful in your teaching?

The answers, of course, cannot be a simple yes or no. These are a few thought questions to guide you in your work. They may help you draw some conclusions about the results of your teaching. Whatever you finally decide, you will probably feel that it was worthwhile doing. Why not try it?

EDITOR'S NOTE. Mrs. Clark learned her arithmetic largely by rote and her original training to teach was largely of this same pattern. She has learned to experiment in her teaching and asks all of us to try newer approaches to learning with the aim of helping children to see the sense of what they are learning and to understand. She has proposed a development from concrete to abstract and with the grouping of children in the class so that the necessary time may be spent on the developmental stages. We may not all want to use her stages of development but let us not be afraid to try new approaches and let us do our best to help pupils think and discover and master the essential learning instead of merely memorizing what someone else has learned. This is a significant difference of attitude toward learning and one that will provide interest and stimulus for succeeding.